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AN INTRODUCTION
TO
PROJECTIVE GEOMETRY

To Mr. T. L. WREN, M.A.,

LATE FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE,
READER IN GEOMETRY IN THE UNIVERSITY OF LONDON.

My dear Wren,

You once said to me, in conversation, that you had been "brought up" on the original edition of this book. Without your friendly and unselfish help I could never have accomplished the task of warming up your infant food, however imperfectly. I hope you will not be entirely displeased with this result of our co-operation, and in that hope I venture to dedicate to you the following pages.

Yours most gratefully,

L. N. G. FILON.

June 1935.

AN INTRODUCTION
TO
PROJECTIVE GEOMETRY

BY

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LONDON
EDWARD ARNOLD & CO.

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516.57
F48i

FIRST PUBLISHED IN 1908
SECOND EDITION . 1916
THIRD EDITION . 1921
FOURTH EDITION . 1935

MADE AND PRINTED IN GREAT BRITAIN BY
WILLIAM CLOWES AND SONS, LIMITED, LONDON AND BECCLES.

PREFACE TO THE FOURTH EDITION

During the quarter of a century which has elapsed since the first edition of this book was published, Projective Geometry has found new practical applications. In particular, the uses of photography in Air Surveying, as well as in Astronomy, involve the principles and methods of projection; and it appears probable that, in both cases, the advantages of graphical constructions will be increasingly appreciated. Meanwhile the older applications to Cartography, Geometrical Optics and Engineering Drawing have lost nothing of their importance.

No apology is therefore needed for the insistence on drawing-board constructions, which was a feature of the earlier editions. Indeed this has been emphasized, in the present edition, by the addition, at the end of all the later chapters dealing with the geometry of the plane, of a set of drawing examples marked *B*, which had previously been restricted to the first seven chapters. From the purely didactic standpoint, actual drawing is even more valuable to clear up difficulties in the more advanced work than it is in the elementary parts of the subject.

It still remains true, however, that the chief interest of projective methods is for the pure mathematician, for whom they provide an instrument of remarkable range and power.

The general scheme of the original edition has remained, save in one important respect, substantially unaltered. In particular I have not modified the lines on which the subject is introduced in Chapter I, though I have tried to remove certain obscurities and have kept the graphical constructions concentrated towards the end of the chapter, so that they may be omitted by those who attach no importance to such constructions. I am aware that this will not satisfy certain critics, but I could not have met their objections without abandoning a conception of the genesis of the subject which I still believe to be the right one.

The chief alteration involving the geometry of the plane has been a rearrangement of order, which brings in Involution before, instead of after, the discussion of foci and focal properties of the conic.

This change was always desirable, for the introduction of foci by means of the focal spheres was never really the natural approach and had the defect of masking the true significance of foci from the projective point of view. The ban on the early introduction of Involution, which used to be imposed by certain University syllabuses, has now been generally abandoned, and the treatment of the whole subject gains thereby in clearness and coherence.

The above modification of plan has necessitated a good many consequential alterations. Chapter VI now deals with ranges and pencils of the second order and self-corresponding elements, and this naturally leads to a discussion of Involution in Chapter VII, followed by the focal properties of the conic in Chapter VIII.

Up to this point the whole treatment, although capable of interpretation in a wider sense, is based upon real elements and constructions actually possible on the drawing-board, as in my view this is essential to give confidence to the beginner.

Chapter IX then introduces imaginary elements and the circular points at infinity. For this, appeal is made, as in the original edition, to algebraic considerations. Although, in strictness, such considerations are outside pure geometry, they are found, in practice, sufficiently convincing to the student, they avoid the usually long-winded arguments based upon a purely geometrical theory of imaginary elements, and it would seem pedantic, at this stage, not to use them. For the same reason I have not hesitated to employ such considerations whenever they lead, as in the treatment of homographic fields or the intersections of loci of various degrees, to general principles most obviously and directly. But I have tried consistently to preserve a geometrical spirit throughout, so far as possible.

Chapters X and XI are devoted to a discussion of homography and reciprocation respectively, in relation to plane fields. In Chapter XI an investigation of Inversion has been added; this is a new feature: although Inversion is not really included under projective methods, it is closely allied to them and usually associated with them in University syllabuses. It is also important to make the student aware of the fact that all one-one point transformations are not necessarily homographic.

Chapters XII–XIV follow the same lines as Chapters XI–XIII in the first edition. The discussion of quadrics, originally limited to one chapter, has appeared inadequate, even at this elementary stage, and has been expanded, so that two chapters, XV and XVI, have now been given to it.

Apart, however, from alterations of order, a large number of improvements and additions have suggested themselves in the course of revision. Among these may be mentioned a new treatment of the circle of curvature in Chapter V, based upon the perspective transformation, in Chapter III, of conics having three-point and four-point contact, and some elementary results on curvature of twisted curves and of quadrics in Chapters XV and XVI; the harmonic envelope and locus of two conics as an illustration of homographic involutions in Chapter XII; an introduction to the general plane cubic and quartic obtained from pencils of conics in Chapter XIII; the focus and directrix property of the sphero-conic in Chapter XIV; a three-dimensional analogue to the complete quadrilateral and quadrangle, and brief discussions of (i) homographic spaces in three dimensions, (ii) inpolar and outpolar quadrics, in Chapters XV and XVI.

Indeed, very few chapters have survived without drastic alteration, and many have been practically rewritten.

A number of new examples have been added; not only have new sets of drawing examples been inserted at the end of Chapters VII, IX–XIII, but a new departure has been to distribute many examples in the text of the chapters, where they serve as illustrations to the articles to which they are appended. In this way the text provides a clue to the solution; conversely the examples help towards the immediate understanding and elaboration of the text. It will be found that the loss of such examples from the sets at the end of the chapters has generally been more than made good, so that in fact the total number of examples in the book has been increased from 406 to 893.

Something may be said about the notation. On the whole, experience shows that the notation employed in the earlier editions has proved workable. Certain improvements in nomenclature, however, have been adopted in the present volume. Thus elements not at infinity have been described shortly as "accessible." "Axis of collineation" has been discarded in favour of the now more usual "axis of perspective." The cumbrous terms "harmonically circumscribed to" and "harmonically inscribed in" have been replaced by "outpolar" and "inpolar." The notion of "field" has been used, in preference to that of "figure," in dealing with general transformations. The use of the term "base" has been generally applied to those elements connected with a geometric form which remain constant; thus a flat pencil has two bases, its vertex and its plane, and the word "cobasal" implies that both these bases are the same. In like manner the quadrangle to which a pencil of conics are circumscribed is referred to as the base of the pencil. I have retained the term "equi-anharmonic" to signify forms such that two corresponding sets of four elements have the same cross-ratio; a modern school of thought uses this term to denote a set of four elements such that they are projective with themselves, when any three of them are interchanged cyclically; but a word is required for equi-anharmonic in the old sense, apart from "projective" or "homographic" which, although ultimately equivalent, proceed originally from a different concept.

It must be admitted that, in many respects, the accepted nomenclature of the subject has not always been happy. The word "sheaf," used in the older books for a set of lines and planes passing through a point, does not really convey to the mind a picture of what is intended, and, indeed, would be more appropriately applied to what is known as a regulus. I have adopted the word "star" instead of "sheaf," following a practice which is gradually being introduced. The term "axial pencil" also seems to me unfortunate, and, in fact, in the geometry of the "star," where flat and axial pencil correspond to range and flat pencil respectively in the plane, actually misleading. A new word is needed for a form consisting of planes, *e.g.* some such word as "fold"; were "fold" used to describe an axial pencil, a "fold" of the second order would denote the set of tangent planes to a cone of the second order, a form for which there is at present no satisfactory short word; conical pencil must obviously denote a cone of lines and corresponds to a range of second order; the cone of planes corresponds to a pencil of second order, but the word "pencil" cannot be used again. Another advantage of the introduction of the word "fold" would be that (leaving systems of conics out of account), a range would always consist of points, a pencil always of straight lines, and a "fold" always of planes. "Axial pencil" is, however, so well entrenched in current practice that I have not ventured to displace it, and I have introduced the word "wrap" to describe, when necessary, the set of tangent planes to a cone.

The term "self-polar," when applied to quadrangles and quadrilaterals, has been changed to "polar"; the nomenclature of the earlier editions appeared unsatisfactory, since such quadrangles and quadrilaterals are not polar figures of themselves. Corresponding changes have been made when dealing with the star and with three-dimensional geometry; also, following Reye, a distinction has been drawn between "polar" and "conjugate" lines with respect to a quadric; in the previous editions the two terms had been used as synonymous.

I have retained the words "pencil of conics (or quadrics)" and "range of conics (or quadrics)," although the latter hardly satisfies me as a description. These terms are by now well-established, and the alternatives would be: either to introduce entirely new words, such as "loop" for "pencil" (suggesting a number of paths through fixed points), and "slide" for "range" (suggesting a deformable curve sliding on fixed guides); or to employ the words "net" and "web," which I have used for linear systems of any grade, to mean, when not accompanied by any qualification, the net and web of the first grade, instead of, as now, those of the second grade. On the whole, however, it seemed that continual changes of notation were to be deprecated. It will be noticed that the word "web" is still used to denote a *tangential* system, and is correlative to "net." I have not followed a practice sometimes adopted, of using "web" to denote a net of the third grade.

The use of Q to denote a quadric has been discontinued, so that the rule that an italic capital always stands for a point, a small italic for a straight line or curve, and a small Greek letter for a plane, surface, or plane field, has now been made universal, with the exception of (i) the circular points at infinity in the plane, which are invariably denoted by Ω, Ω', (ii) the circle at infinity, for which the notation $\odot$ has been introduced. With the exception (i) just noted, Greek capitals are used to denote three-dimensional aggregates or fields, when such enter into consideration.

My thanks are due to the authorities of the University of London and of University College, London, and also to the Syndics of the Cambridge University Press, for permission to include in the examples a number of questions taken from London and Cambridge examination papers.

I owe a specially heavy debt of gratitude to my friend and colleague, Mr. T. L. Wren, Reader in Geometry in the University of London, University College, for his invaluable help and suggestions throughout. Mr. Wren very kindly undertook the laborious

task of looking over the whole MS. of the revised work and has actually checked practically every example, and suggested many new ones. It is not too much to say that, but for his devoted help, it would have been impossible to complete the revision of the book in the very limited time at my disposal. Many of the changes in nomenclature are based on his suggestions.

Finally, I have to express my thanks to Mr. F. P. Dunn and to the staff of Messrs. Edward Arnold & Co. for their uniform courtesy and assistance and for the care they have taken in the preparation of the text and diagrams of the present edition.

L. N. G. F.

June 1935.

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PROJECTIVE GEOMETRY

CHAPTER I

SPACE AND PLANE PERSPECTIVE

1. Plane figures in space perspective. Let there be a set of straight lines or rays, all passing through a point V; these lines are not limited to lie in a plane. They form what is known as a **star** of lines, of which V is said to be the centre or **vertex.**

If we now cut such a star by two planes, α_1, α_2 (Fig. 1), so that P_1, Q_1, etc., are the intersections of the rays with α_1, and P_2, Q_2, etc., are the intersections of the same rays with α_2, we obtain two sets of points which form corresponding figures in the two planes.

Two such figures are said to be in **space perspective.** If we place the eye at V, the two figures appear to cover one another, since the lines joining corresponding points pass through V. The process by which we pass from one figure to the other is termed **projection,** and V is spoken of as the **vertex of projection**; the figure in α_1 is said to be projected from V upon α_2 into the figure in that plane, and the second figure is said to be the projection from V of the figure in α_1. The plane upon which we project is referred to as the **plane of projection.**

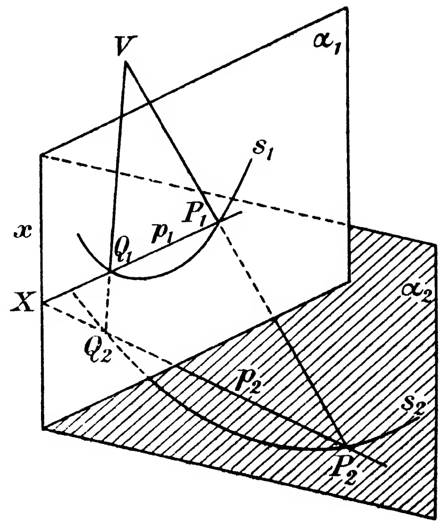


FIG. 1.

It should be noted that the two planes α_1, α_2 need not be, as shown in Fig. 1, on the same side of the vertex V. If a photograph of a plane diagram, or an air-photograph of flat ground, be taken, it is known from the laws of geometrical optics that the lines joining any point of the object to the corresponding point of the photograph all pass through a point known as the centre of the object-glass. Such a photograph is therefore a projection of the object upon the photographic plate.

It will be convenient, in what follows, when it is desired to discriminate between the figure projected and its projection, to refer to the first as the **field** and to the second as the **picture.** An element of the field may then be described as an **object,** and its projected element as its **image.**

It must be remembered, however, that this distinction is artificial, and introduced merely for convenience of description, for geometrically the relation between field and picture is interchangeable, and either figure may be regarded as the projection of the other.

Projective Geometry studies the relations of corresponding figures obtained by this process. It will be found that certain important properties are transmitted unaltered, so that theorems involving only such properties hold of both the original and the projected figure.

Although the subject will here be treated from the point of view of theoretical geometry, the reader should bear in mind that it has many important applications in practice to engineering drawing, interpretation of photographs, cartography and astronomical photography.

2. Notation. The study of this subject is greatly facilitated by the use of the following systematic notation, which will be adhered to throughout.

The points, straight lines and planes which enter into the construction of a geometrical figure will be spoken of as its **elements.** The use of this general term will often enable us to state results which hold equally of certain sets of points, or straight lines, or planes, without specifying explicitly the particular type of element considered.

Points will be denoted by italic capitals $A, B, C, \ldots$.

Straight lines, and also curves, will be denoted by small italic letters $a, b, c, \ldots$; planes, and also other surfaces, by small Greek letters $\alpha, \beta, \gamma, \ldots$.

Two elements are said to be **incident** if one lies in or passes through the other.

Thus if A lies on a, then a, A are incident.

If α contains a, then α, a are incident.

When two symbols are combined in the form of a product, the joint symbol denotes that element which is incident with the original two.

Thus AB denotes the straight line passing through the points A, B; Aa denotes the plane determined by the point A and the line a; $\alpha\beta$ denotes the line of intersection of the planes α, β; $\alpha\beta\gamma$ is the point of intersection of the three planes α, β, γ.

Such a joint symbol is not always interpretable. Thus ab has no meaning if a, b are lines in space which do not intersect. If, however, a, b intersect, there are two possible meanings for ab, namely, the point of intersection of a and b, or the plane determined by a and b. When dealing with problems in a plane, the first interpretation will always be adopted. In other cases the ambiguity will be removed by the use of the word " point " or " plane " before the symbol ab.

The straight line joining two points will be described briefly as their **join**; the intersection of two straight lines or planes as their **meet.**

In dealing with corresponding figures, corresponding elements will invariably be *lettered* alike, the figures to which they belong being indicated by suffixes or by accents. Thus A_1 corresponds to A_2, a_1 to a_2 and so on. The student should be very careful to adhere rigidly to this practice, as random lettering obscures the correspondence of elements, which is their significant property and should be brought into prominence by every possible means.

The student is supposed familiar with the notion of a segment on a straight line as having *sense*, as well as magnitude. In this connection it should be noted that the sense of a segment will be indicated by the order of naming the letters.

Thus
$$AB = -BA,$$
and, whatever be the order of the points A, B, C on the line
$$AB + BC = AC.$$

When it is desired to consider merely the length of a segment AB, this will be written length AB or more shortly $|AB|$.

When the symbol AB is used it will in general be evident from the context whether the infinite straight line AB is meant, or only the segment AB.

The intersection of a straight line or a plane with a given plane will sometimes be referred to as the **trace** of that line or plane on the given plane.

3. Corresponding lines and curves in projection. Collineation. Returning now to the two figures in space perspective (Fig. 1), let P_1 trace out a straight line p_1 in the field; then its image P_2 will trace out a straight line in the picture.

For the ray VP_1 sweeps out the plane π determined by V and p_1; since the points P_2 lie on the rays VP_1, they all lie in π, and therefore on the meet of π and the picture plane, which is a straight line p_2; p_2 is the projection, or image, of p_1.

Notice that, if P_1 lies on p_1, then P_2 lies on p_2; thus properties of *incidence* are preserved in projection.

Similarly P_1 may describe any curve s_1 in the field; P_2 then describes the corresponding curve s_2 in the picture, and s_1, s_2 are sections of the same cone, whose vertex is V, by the field and picture-planes, respectively; such a cone, of course, is not restricted to be right circular.

If P_1, Q_1 are two object-points, p_1, the line joining them, then, by what has already been stated, P_2 and Q_2 both lie on p_2. Thus joins of corresponding pairs of points are corresponding lines. Similarly meets of corresponding pairs of lines are corresponding points.

If Q_1 moves up to P_1 along the curve s_1, Q_2 moves up to P_2 along the curve s_2; p_1, p_2 then approach the tangents to s_1, s_2 at P_1, P_2 respectively, while still remaining corresponding lines.

Thus a tangent and its point of contact in the field project into a tangent and its point of contact in the picture, so that properties of *tangency* are preserved in projection.

It should also be noted that the correspondence is not limited to such points or lines of the figures as are actually present in the diagram, or under immediate consideration, but involves potentially all possible points and lines of either plane. Any element whatever in the field plane may be selected and its corresponding element in the picture constructed; and conversely.

We shall frequently have occasion to refer to figures (either in the same or in different planes), whose points correspond in such a manner that the points of a straight line in either figure correspond to the points of a straight line in the other. Two such corresponding figures will be referred to briefly as being **in collineation,** or as forming a collineation. It is clear from the above that a figure and its projection on any plane are a particular case of a collineation.

4. Elements at infinity. Let a plane σ (Fig. 2) be now drawn through the vertex of projection V parallel to α_2. It will meet the plane α_1 in a line i_1 which is parallel to α_2. Also if I_1 is any point of i_1, VI_1 is parallel to α_2.

Similarly, if the plane τ be drawn through V parallel to α_1, to meet α_2 in a line j_2, which is parallel to α_1, and J_2 is any point of j_2, VJ_2 is parallel to α_1.

According to the language of Euclidean Geometry, the line VI_1 does not meet α_2, and VJ_2 does not meet α_1, so that I_2, J_1 cannot be found in this case, nor can the lines i_2, j_1 be constructed.

In order to avoid the complications which would continually result from the necessity of considering such cases of exception, we introduce, by a convention, a set of new ideal elements, points, lines and plane, which are called the **elements at infinity.** By means of these elements the cases of exception are removed, and theorems can be stated in a more general manner.

We shall say that a given direction in space determines one **point at infinity,** through which all straight lines parallel to this direction are supposed to pass.

This gives a construction for the line joining P to a given point at infinity, viz. draw the parallel through P to the direction defining that point at infinity.

Further, all planes parallel to a given plane are to be regarded as intersecting in one straight line, which will be called the **line at infinity** in that plane.

Any line lying in a plane has its point at infinity on the line at infinity in that plane.

From these definitions it follows that the aggregate of all points at infinity is met by any straight line in one point, and by any plane in one straight line. It possesses therefore the essential characteristics of a plane, and will be spoken of as the **plane at infinity.**

The student should note carefully that on any line there is *one* point at infinity only, not *two.* For if there were two points at infinity, a parallel to the line would pass through both of them, and two straight lines would intersect in more than one point, which would violate a fundamental postulate.

He may convince himself of the identity of the two opposite infinities on a line by imagining a ray through a point O outside the line and meeting the line at P to rotate continuously about O. P travels continuously along the line until the rotating ray passes through the position of parallelism, when P suddenly passes from one extremity of the line to the other, showing that these opposite infinities are not separated.

To call attention to the fact that an element lies at infinity, the symbol ∞ will be used as an index, thus A^{∞}, a^{∞}, etc.

We shall therefore, from now on, draw no distinction between pairs of lines, or planes, which intersect at a finite distance, and pairs which are parallel. In every case, a point or line of intersection will be assumed, but, if the elements are parallel, their intersection will be at infinity.

It will, however, sometimes be convenient, for the sake of brevity,

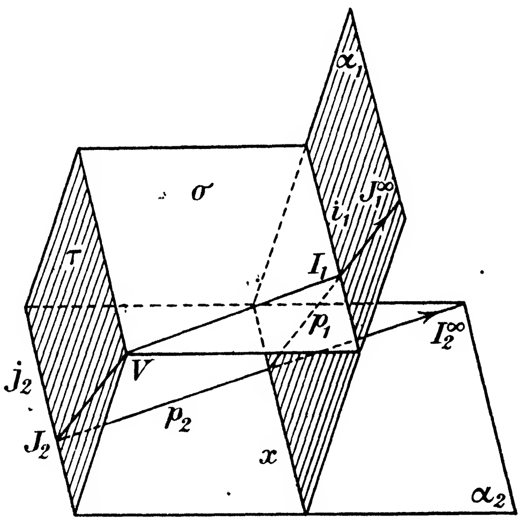


FIG. 2.

to use a single word to specify that an element does not lie wholly at infinity. Such an element will be said to be **accessible.**

5. Vanishing points and lines. We are now able to complete our correspondence between the field and the picture.

For if on any line p_2 (Fig. 2) of the picture plane α_2 we take the point $I_2{}^{\infty}$ at infinity, $VI_2{}^{\infty}$ is parallel to p_2 and therefore to α_2. It thus lies in the plane σ and meets α_1 at a point of i_1, namely I_1, which corresponds to $I_2{}^{\infty}$. Accordingly all the points at infinity of α_2 correspond to points of i_1. Conversely, if I_1 is a point of i_1, VI_1 is parallel to α_2 and meets α_2 at a point at infinity. There is thus a complete correspondence between the points of i_1 and the points at infinity of the plane α_2, which justifies the statement of Art. 4, that the points at infinity of a plane are to be regarded as lying on a straight line. The line at infinity of α_2, which corresponds to i_1, will be denoted by $i_2{}^\infty$ and is determined as the meet of α_2 with Vi_1, that is, with σ.

Similarly if $J_1{}^\infty$ is the point at infinity on any line p_1 in α_1, its corresponding point J_2 lies on the line j_2 of Fig. 2, and VJ_2 is parallel to p_1. The points at infinity of α_1 lie on the straight line $j_1{}^\infty$ which is the meet of τ and α_1.

The line i_1 of α_1, which corresponds to the line at infinity of α_2, is termed the **vanishing line** of α_1. Similarly j_2 is the vanishing line of α_2.

In like manner a point I_1 of p_1, which lies on i_1, and therefore corresponds to the point at infinity $I_2{}^\infty$ of p_2, is termed the **vanishing point** of p_1, and the point J_2, where p_2 meets j_2, and which corresponds to $J_1{}^\infty$ on p_1, is termed the vanishing point of p_2.

An important result follows immediately.

Since VI_1 is parallel to p_2, the line joining the vertex of projection to the vanishing point of any line is parallel to the corresponding line.

Hence the angle subtended at V by the vanishing points of two lines (say, p_1, q_1) is equal to the angle between the corresponding lines p_2, q_2.

6. Axis of perspective. We have seen that, if p_1, p_2 are two corresponding lines in any two figures in space perspective, they are sections by the field plane α_1 and the picture plane α_2 of the same plane π through the vertex of projection. They are therefore coplanar lines and must intersect at a point X (Fig. 1). This point X is common to α_1 and α_2 and therefore lies on the intersection x of these two planes.

Thus corresponding lines intersect on this line x, which is called the **axis of perspective.**

In particular the vanishing line i_1, and the line $i_2{}^\infty$ at infinity in α_2, meet on x, or x and i_1 meet at infinity, that is, are parallel.

Similarly j_2 and x are parallel.

Hence both vanishing lines are parallel to the axis of perspective, which is otherwise evident from consideration of Fig. 2.

Examples

1. Prove that, if two figures are in space perspective, points on the axis of perspective are self-corresponding.

2. Show that lines parallel to the axis of perspective correspond to lines parallel to the same direction.

3. Prove that, in any projection, there are two points in the field such that every angle at either point projects into an equal angle in the picture.

4. Show that the two points in Ex. 3 subtend a right angle at the vertex of projection.

5. Show that, in two figures in space perspective, there are two corresponding lines, parallel to the axis of perspective, such that any segment on one line corresponds to an equal segment on the other.

6. Prove that the two lines in Ex. 5 are symmetrical to the axis of perspective with respect to the two vanishing lines.

7. Preliminary proposition on intersecting lines. We now prove the following proposition: if a set of lines in space are such that any one line intersects every other, the lines must either pass through a point, or lie in a plane.

Let a, b be two lines of the set, V the point ab, π the plane ab.

If now all the lines of the set do not lie in π, let c be a line which does not lie in π. Then the only point where it can meet both a and b is V, and therefore c passes through V.

Let d be any other line of the set. It must meet a, b and c. If it meets them at points A, B, C other than V, then, since ABC is a straight line, VA, VB, VC, that is a, b, c, must lie in a plane, or c lies in π, which we have assumed not to be the case. Hence d must pass through V, and therefore every line of the set passes through V.

Thus all the lines of the set either lie in π, or pass through V, which proves the proposition.

8. Intersection of corresponding lines a sufficient condition for space perspective. If now we are given two corresponding figures in different planes α_1, α_2, which are in collineation (Art. 3), and which have the property that any two corresponding lines intersect (which necessarily happens on the meet x of α_1, α_2), then the figures are in space perspective.

For let P_1, Q_1 be any two points of the figure in α_1; P_2, Q_2 the corresponding points of the figure in α_2. P_1Q_1, P_2Q_2 are corresponding lines; by hypothesis they intersect and the four points P_1, Q_1, P_2, Q_2 lie in a plane. Hence the lines P_1P_2, Q_1Q_2, which join corresponding points, lie in a plane, and therefore intersect.

It follows that every join of corresponding points intersects every other such join; hence by Art. 7 these joins either all pass through a point V, or lie in a plane π. But they cannot do the latter, for in this case π would contain all the points of both figures, and the planes α_1, α_2 would coincide.

Hence P_1P_2, Q_1Q_2, etc., all pass through a vertex V and the figures are in space perspective.

9. Rabatment. Figures in plane perspective. The field and its picture can best be compared, and, if drawing-board constructions are required, must be compared, by bringing them into the same plane.

The method of doing this which is the most convenient is to turn the plane of the field about the axis of perspective, carrying its figure with it, until it comes into the picture plane. Alternatively the picture can be rotated about the axis of perspective into the field plane.

This procedure of rotating one plane figure into another plane about the line of intersection of the planes is termed **rabatment,** and we are said to **rabat** one figure into the plane of the other.

Suppose now we start with two figures ϕ_1, ϕ_3 in space perspective, and we rabat ϕ_3 upon the plane of ϕ_1, so that it becomes a figure ϕ_2, ϕ_2 and ϕ_3 are, of course, congruent or superposable, but are in different positions and will be considered distinct figures.

The rotation, however, has not affected the axis of perspective, and, after rabatment, the corresponding lines of ϕ_1 and ϕ_2 still meet on x.

ϕ_1 and ϕ_2 are now corresponding figures in the same plane which have the property that corresponding lines meet on a fixed line x of the plane.

Conversely any two corresponding figures ϕ_1, ϕ_2 in collineation, which lie in a plane and are such that corresponding lines meet on a fixed line x, may be obtained by rabatment from two figures in space perspective.

For rotate ϕ_2 about x out of the plane into a new position ϕ_3, then by Art. 8, ϕ_1 and ϕ_3 are in space perspective, and ϕ_2 is obtained from ϕ_3 by rabatment.

Two such figures are said to be **in plane perspective,** or **in homology** or **homological.** The axis x is sometimes called the **axis of homology ;** we shall continue to use the term axis of perspective.

As the figures are in the same plane, a new consideration now arises, to which the reader must, from now on, pay very careful attention, namely, that the *same* point or line of the plane may have an entirely different significance, according as we treat it as belonging to one figure or to the other. This we shall indicate by denoting such a point or line by a different letter in the two cases.

For example, the line at infinity in the plane may be denoted by either $i_1{}^\infty$ or $j_2{}^\infty$ according as we regard it as line of ϕ_1 or of ϕ_2. In the first case its corresponding line in ϕ_2 is the vanishing line i_2, which is the rabatment of i_3; in the second case its corresponding line in ϕ_1 is the vanishing line j_1.

We see that there are two vanishing lines in the plane, one for each figure. Note that the vanishing line of ϕ_1, say i_1, if treated as a line, say p_2, of ϕ_2, has no special significance.

10. Pole of perspective. Let ϕ_1, ϕ_2 be two figures in plane perspective in a plane α, and let x be their axis of perspective (Fig. 3).

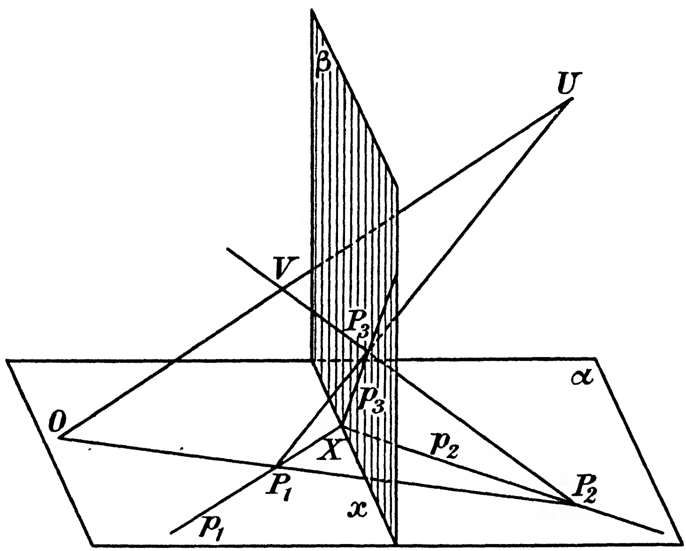


Fig. 3.

Through x draw any plane β, and take any point U in space, not lying in α or β.

Project ϕ_1 from U on to β. This gives a figure ϕ_3 in space perspective with ϕ_1, x being again the axis of perspective.

ϕ_2 and ϕ_3 are then corresponding figures in different planes, α and β, such that corresponding lines intersect. Hence by Art. 8 they must be in space perspective. Let V be the vertex of projection for ϕ_2 and ϕ_3. Join UV meeting α at O.

Now if P_1, P_2, P_3 are any set of corresponding points in the three figures, P_1U and P_2V meet at P_3 and are therefore coplanar lines. Therefore the four points U, V, P_1, P_2 lie in one plane, and UV

meets P_1P_2. But since P_1P_2 lies in α, and UV does not (for U was taken outside α), P_1P_2 can only meet UV at the point O of UV which lies in α. Hence P_1P_2 passes through O, which is a point independent of the choice of the points P_1, P_2, since U, V do not depend on P_1, P_2. Thus we arrive at the result that: the joins of corresponding points of two figures in plane perspective pass through a fixed point O of the plane, which is called the **pole of perspective.**

EXAMPLES

1. Prove that two figures in plane perspective can always be derived as projections of the same figure from two different vertices.

2. Show that, in a plane perspective, points on the axis and lines through the pole of perspective are self-corresponding.

3. Two curves are in plane perspective. Show that the axis of perspective must be one of their common chords, and the pole of perspective must be one of the intersections of their common tangents.

4. Show that the property of Art. 6, Ex. 3 holds equally of figures in plane perspective.

5. Show that the properties of Art. 6, Exs. 5 and 6 hold equally of figures in plane perspective.

6. Prove that if, in a plane perspective, a curve passes through the pole O of perspective, it touches its corresponding curve at O.

11. Desargues' perspective triangle theorem. Let there be in a plane two corresponding triangles $A_1B_1C_1$, $A_2B_2C_2$ (Fig. 4), and let $B_1C_1=a_1$, $B_2C_2=a_2$, etc.

If the triangles are such that a_1a_2, b_1b_2, c_1c_2, which we will denote shortly by X, Y, Z respectively, are collinear, then it follows at once from Arts. 9 and 10 that the triangles are in plane perspective, and therefore A_1A_2, B_1B_2, C_1C_2 pass through a point O.

The converse of this is an important theorem, namely, that, if A_1A_2, B_1B_2, C_1C_2 are concurrent, then a_1a_2, b_1b_2, c_1c_2 are collinear.

Join XY and denote it by x. Let c_1 meet x at Z. Join ZA_2 and, if it do not coincide with c_2, let $ZA_2=c_2'$, meeting a_2 at B_2' (Fig. 4). Then the triangles $A_1B_1C_1$, $A_2B_2'C_2$ are such that the meets of corresponding sides a_1a_2, b_1b_2, c_1c_2', *i.e.* X, Y, Z, are collinear. Therefore, by the theorem just proved A_1A_2, B_1B_2', C_1C_2 are concurrent, and B_1B_2' passes through the meet O of A_1A_2, C_1C_2. But we are given that B_1B_2 also passes through O. Hence B_1B_2 and B_1B_2' coincide. Thus B_2 and B_2' coincide, since each is the meet of the same two lines a_2 and OB_1. Hence A_2B_2, that is c_2, passes through Z, and a_1a_2, b_1b_2, c_1c_2 are collinear.

If now we have two figures ϕ_1, ϕ_2 in a plane, which form a collineation and which possess the property that all joins of corresponding points pass through a fixed pole O, then all meets of corresponding lines must lie on a fixed axis x.

Take two pairs of corresponding lines a_1, a_2; b_1, b_2, and let $X = a_1a_2$, $Y = b_1b_2$. Consider any other corresponding lines c_1, c_2 such that a_1, b_1, c_1 do not pass through a point. The triangles formed by $a_1b_1c_1$, $a_2b_2c_2$ have the joins of corresponding vertices passing through O, and therefore c_1c_2 (or Z) lies on XY. But c_1 is arbitrary, except that it must not pass through a_1b_1.

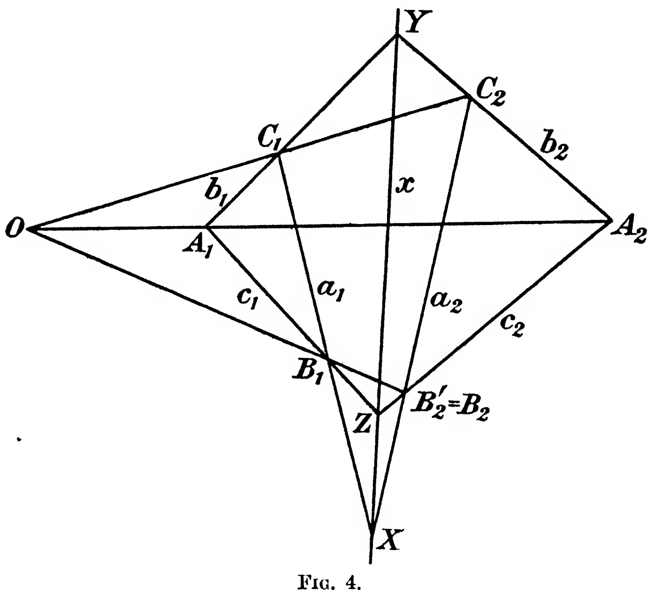


Fig. 4.

If now d_1 is a line through a_1b_1, it is always possible to form a triangle with d_1 and either b_1, c_1 or a_1, c_1. Applying again Desargues' theorem d_1d_2 lies on YZ (or XZ), that is, on x. Thus all pairs of corresponding lines meet on x without exception, and the property of the pole of perspective is a sufficient, as well as a necessary, condition that two figures in collineation are in plane perspective.

Example

Prove directly, without quoting the results of Arts. 8–10, that if $A_1B_1C_1$, $A_2B_2C_2$ are two triangles in a plane and A_1A_2, B_1B_2, C_1C_2 are concurrent, then a_1a_2, b_1b_2, c_1c_2 are collinear; and conversely.

12. Cylindrical projection. Rabatment as a projection. If the vertex of projection is at infinity, as V^∞, the lines P_1P_2, Q_1Q_2, etc., joining corresponding points of the field and picture, all pass through V^∞, and are parallel. The projection is said to be **cylindrical,** two corresponding curves being sections of a cylinder; whereas, in the case where the vertex is accessible, corresponding curves, as explained in Art. 3, are sections of a cone, and this kind of projection is sometimes termed **conical.**

In cylindrical projection $V^\infty j_1{}^\infty$ is the plane at infinity and meets the picture plane in the line at infinity of that plane, which is accordingly $j_2{}^\infty$. Thus lines at infinity correspond, and the vanishing lines are themselves at infinity. Hence two parallel lines in the field, which meet at a point $J_1{}^\infty$, correspond to two lines which also meet at a point $J_2{}^\infty$, that is, to two parallel lines in the picture.

If V^∞ is in the direction perpendicular to the picture plane, so that all joins P_1P_2 are perpendicular to that plane, we have the special case of cylindrical projection known as **orthogonal projection.** The picture is then said to be the orthogonal projection of the field.

We will now show that rabatment, as defined in Art. 9, is equivalent to a cylindrical projection.

Let ϕ_3, a figure in a plane β, be rabatted upon a plane α into a figure ϕ_2. If P_3 is any point of ϕ_3, and P_3N is drawn perpendicular to x, where $x=\alpha\beta$, then, during rabatment, P_3 describes a circular arc P_3P_2 with N as centre, in a plane perpendicular to $\alpha\beta$.

Then P_3P_2 is perpendicular to the plane γ which bisects the dihedral angle between α, β, through which the rotation takes place. The figures ϕ_3, ϕ_2 are therefore in space perspective from the point U^∞ at infinity in the direction perpendicular to γ, that is, ϕ_2 is derived from ϕ_3 by a cylindrical projection.

Consider now a third figure ϕ_1, lying in α, and in plane perspective with ϕ_2, with x as axis of perspective.

Since corresponding lines of ϕ_1, ϕ_2, ϕ_3 meet on x, ϕ_1 and ϕ_3 are in space perspective from some vertex V (Art. 8). To find V take for the plane of the paper in Fig. 5 the plane perpendicular to x passing through the pole O of perspective for ϕ_1, ϕ_2. This plane contains U^∞.

Let Fig. 5 represent an elevation in this plane, so that all points of x appear as a single point X; in a similar manner, if the vanishing lines i_1, j_2, j_3 meet this plane at I_1, J_2, J_3, these lines, being parallel to x, are shown in the figure by their traces I_1, J_2, J_3.

By the property of the pole of perspective, I_2^{∞} is at infinity on OI_1. $U^{\infty}I_2^{\infty}$ is the line at infinity in the plane of the paper and meets β at I_3^{∞}, the point at infinity in the trace of β on the diagram. $I_1I_3^{\infty}$ is therefore parallel to this trace.

Similarly J_2 corresponds to J_1^{∞} on OJ_2, and $XJ_3 = XJ_2$, by

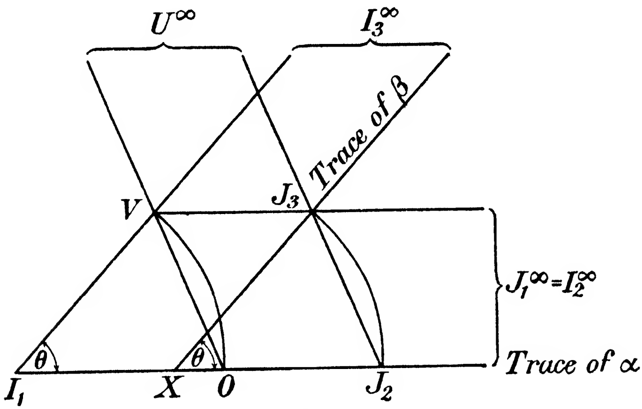


FIG. 5.

rabatment. Thus $J_3J_1^{\infty}$ is parallel to the trace of α. $J_3J_1^{\infty}$ and $I_3^{\infty}I_1$ meet at V. Also by Art. 10, $U^{\infty}V$ meets the trace of α at O.

Clearly XI_1VJ_3, OVJ_3J_2 are parallelograms.

Hence $$XI_1 = J_3V = J_2O \qquad (1)$$

Adding I_1J_2
$$XJ_2 = I_1O \qquad (2)$$

Thus the distance of a vanishing line from the axis of perspective is equal to the distance of the pole of perspective from the other vanishing line, both distances being measured with proper sign.

But further, $XJ_2 = XJ_3$, by rabatment; and $XJ_3 = I_1V$, being opposite sides of a parallelogram.

Hence $$I_1O = I_1V \qquad (3)$$

also I_1V is parallel to β.

If, then, one of two figures in plane perspective be rotated about the axis of perspective, the vertex of projection of the resulting figures in space perspective describes a circle obtained by rotating the pole of perspective about the vanishing line of the figure which remains fixed and the angle of both rotations is the same.

Conversely, if two figures are in space perspective from a vertex V, and are brought into the same plane by rabatment, the pole of perspective after rabatment is obtained by rabatting V about the vanishing line of the figure which remains fixed, a construction which will be found useful in drawing examples.

13. Particular cases of figures in plane or space perspective. Several cases of figures in plane perspective will be already familiar to the reader acquainted with elementary Geometry, thus :

(i) **Similar and similarly situated figures in a plane.** Here all corresponding lines are parallel, and meet on the line at infinity, which is accordingly the axis of perspective, x^{∞}. Since this corresponds to itself, both vanishing lines coincide with it. The theorem of Art. 10 then shows that joins of corresponding points pass through a pole O of perspective, which is the usual centre of similitude.

(ii) **Directly congruent figures similarly situated in a plane.** This is a special case of the preceding. Here again, the axis of perspective is at infinity. If corresponding segments are drawn in the same sense, the joins of corresponding points are parallel, and the pole of perspective is at infinity. But if corresponding segments are drawn in opposite senses, the pole of perspective is at a finite distance.

(iii) **Symmetrically congruent figures.** These can be obtained by turning one figure over about a line x of the plane, through two right angles. This gives figures symmetrical about the line x and the process may also be briefly described as **reflexion** in the line x.

In this case corresponding lines meet on x, and the lines joining corresponding points are perpendicular to x. The figures are in plane perspective, x being the axis of perspective, and the pole of perspective being the point at infinity in the direction perpendicular to x.

(iv) **Figures superposable by rotation about a line in their planes.** We have already seen that two congruent figures, derived one from the other by a rotation about the intersection of their planes, are in space perspective.

(v) **Two similar (including congruent) and similarly situated figures in parallel planes** are in space perspective. For corresponding lines, being parallel, intersect at infinity on the meet of the two planes, which is the line at infinity of both. The result then follows from Art. 8.

(vi) **Two figures in a plane derived one from the other by a uniform stretch** give a particular case of plane perspective. A **stretch** is defined as follows: if x, y be two fixed co-ordinate axes in the plane (not necessarily at right angles), the point P_1, whose co-ordinates are (x_1, y_1), corresponds to the point P_2, whose co-ordinates are (x_2, y_2), where $x_2 = x_1$, $y_2 = ky_1$, k being a constant termed the **stretch-ratio.** The transformation thus consists in stretching the ordinate P_1N (Fig. 6) in the ratio k; hence the name. x is termed the axis of stretch, y the direction of stretch.

We first prove that the two figures ϕ_1, ϕ_2 are in collineation, *i.e.* that a straight line locus in ϕ_1 corresponds to a straight line in ϕ_2. To do this, take any line p_1 passing through P_1 and meeting

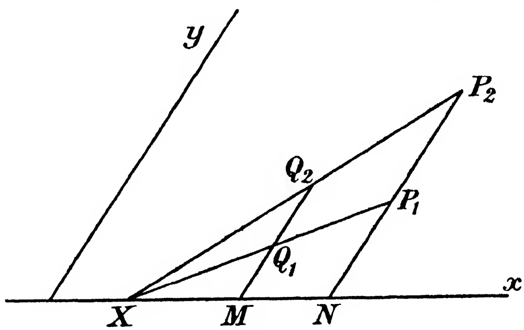


FIG. 6.

x at X. Let Q_1 be any other point of p_1. Join XP_2, meeting the ordinate Q_1M at Q_2. Then by the properties of similar figures $Q_2M : Q_1M = P_2N : P_1N = k$. Hence Q_2 is the point corresponding to Q_1, and the locus of points Q_2 is XP_2, which we call p_2.

Moreover, the joins of corresponding points P_1P_2 pass through a fixed point, namely, the point at infinity on y, and the meets X of corresponding lines p_1, p_2 lie on a fixed line, namely, x. Accordingly ϕ_1 and ϕ_2 are in plane perspective.

EXAMPLES

1. Prove that a stretch is equivalent to the rabatment of a cylindrical projection.

2. A triangle $A_2B_2C_2$ is given as the orthogonal projection of a right-angled isosceles triangle $A_1B_1C_1$, the angle at C_1 being a right angle. If $C_2A_2 = 3$ inches, $C_2B_2 = 4$ inches, and the angle $A_2C_2B_2 = 60°$, and if the axis of perspective passes through C_2 and makes with C_2A_2, C_2B_2 angles of 15° and 45° respectively, construct the rabatted triangle $A_1B_1C_1$.

14. Projective figures. The process of projection may clearly be repeated any number of times. Thus a figure ϕ_1 in a plane α_1 may be projected from some vertex into ϕ_2 in a plane α_2; ϕ_2 may then be projected from a second vertex into ϕ_3 in a plane α_3; and so on. The final figure ϕ_{n+1}, obtained from ϕ_1 after n projections, lies in a plane α_{n+1} which may, or may not, be identical with the original plane α_1.

Two such figures ϕ_1, ϕ_{n+1}, which are derivable one from the other by a finite number of projections, are said to be **projective.** Figures in space perspective are clearly projective, being derivable by a single projection. Figures in plane perspective are also projective, being derivable by two projections (*cf.* Art. 12). But, in general, projective figures are not either in space or in plane perspective.

The projective property is what mathematicians term *transitive*, that is, if ϕ_1 is projective with ϕ_2, and ϕ_2 with ϕ_3, then ϕ_1 is projective with ϕ_3.

For let Σ_1 be the set of projections which transform ϕ_1 into ϕ_2 and Σ_2 the set which transform ϕ_2 into ϕ_3, then Σ_1 and Σ_2 applied in succession form a finite set of projections which transform ϕ_1 into ϕ_3.

15. Particular cases of projective figures. The following are cases of projective figures.

(i) **Figures in plane or space perspective.** This has already been explained in the last article.

(ii) **Coplanar figures superposable by rotation about a point O of this plane.** To prove this, take a figure ϕ_1 in the plane, and two lines x, y through O, also in the plane. Let ϕ_2 be the reflection of ϕ_1 in x and ϕ_3 the reflection of ϕ_2 in y. ϕ_2 and ϕ_1 are oppositely congruent; so are ϕ_2 and ϕ_3. Hence ϕ_1, ϕ_3 are directly congruent figures. The point O is clearly unaltered by the transformation. Hence ϕ_3 is derivable from ϕ_1 by a rotation about O. But the line x of ϕ_1 corresponds to itself in ϕ_2, and the line x of ϕ_2 is turned over about y, that is, it becomes a line x_3 which makes with x twice the angle between x and y. By taking this last angle (which is at our disposal) equal to half the given rotation, we have ϕ_1, ϕ_3 connected in the required manner.

Since ϕ_1, ϕ_2 are in plane perspective (Art. 13 (iii)), they are projective. So also are ϕ_2, ϕ_3. Therefore ϕ_1, ϕ_3 are projective.

(iii) **Any two congruent figures ϕ_1, ϕ_2.** Let them be in different planes α_1, α_2. By rabatting ϕ_1 about $\alpha_1\alpha_2$ into α_2, we obtain two congruent figures ϕ_3, ϕ_2 in the same plane. These can always be made *directly* congruent by rabatting in a suitable sense. This process is equivalent to a projection (Art. 12), so that ϕ_1 and ϕ_3 are projective.

Now rotate ϕ_3 about any point of its plane until corresponding lines are parallel. Let ϕ_4 be the resulting figure. Then by (ii) above ϕ_3 and ϕ_4 are projective.

ϕ_2 and ϕ_4 are now directly congruent and similarly placed. Hence they are in plane perspective by Art. 13, so that ϕ_4 and ϕ_2 are projective.

Combining the above results we see that ϕ_1 and ϕ_2 are projective.

If the figures ϕ_1, ϕ_2 are in the same plane, they may be directly, or oppositely, congruent. If the former, we proceed as from the stage ϕ_3 in the previous case. If the latter, reflect ϕ_1 in a line x which bisects the angle between any pair of corresponding lines. The new figure is now congruent and similarly situated to ϕ_2, and we have ϕ_1 projective with ϕ_2 as before.

(iv) **Any two similar figures** ϕ_1, ϕ_2. Proceeding as in the previous case, we transform ϕ_1 by a series of projective operations into a figure similar to ϕ_2 and similarly situated. A projective transformation (see Art. 13) then transforms the last obtained figure into ϕ_2.

16. Construction of figures in plane perspective from given data. A plane perspective relation is entirely determined when certain elements are given, and we may then construct, point by point, or line by line, the figure ϕ_2 which is in plane perspective with a given figure ϕ_1.

Let the pole O and axis x of perspective be given, and, in addition, *either* a pair of corresponding points A_1, A_2 (whose join must pass through O), *or* a pair of corresponding lines a_1, a_2 (whose meet must lie on x).

One pair of these additional data are immediately derived from the other pair. For if Y is any point on x (Fig. 7), Y is self-corresponding, by the property of the axis of perspective. Hence, if A_1, A_2 be given, YA_1, YA_2 are corresponding lines, which may be taken as a_1, a_2. Conversely, if a_1, a_2 be given, any ray through O meets them at corresponding points A_1, A_2.

If now P_1 is any point of ϕ_1, join A_1P_1 meeting x at X. Then XA_2 corresponds to XA_1, and meets OP_1 at P_2.

Again, if p_1 is any line of ϕ_1, meeting x at X and a_1 at A_1, join OA_1 meeting a_2 at A_2. Then $XA_2 = p_2$.

Alternatively, instead of A_1, A_2, or a_1, a_2, we may be given the vanishing line of one figure, say i_1. If then I_1 is any point of i_1, I_2^∞ is the point at infinity on OI_1, and i_2^∞ is the line at infinity in the plane. These provide a pair of corresponding points and a pair of corresponding lines.

The previous constructions then become :

Join I_1P_1 meeting x at X. XI_2^∞ is the parallel through X to OI_1 and meets OP_1 at P_2.

Let p_1 meet x at X and i_1 at I_1. Then the parallel through X to OI_1 is p_2.

From the last result we see that the line corresponding to p_1 is parallel to the join of O to the vanishing point of p_1. Thus the angle between the lines p_2, q_2 is equal to the angle subtended at the pole of perspective by the vanishing points of p_1, q_1. This will be found to be an important property in constructions connected with such figures. The reader should compare the corresponding result for figures in space perspective at the end of Art. 5.

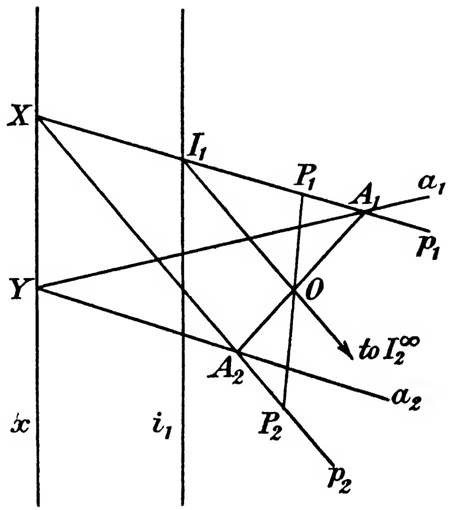


FIG. 7.

EXAMPLES

1. Prove that, when a vanishing line and the pole and axis of perspective are given, the construction given in Art. 16 for the line corresponding to a given line p_1 fails when p_1 is parallel to the axis, and give an appropriate construction in this case.

2. Show that, if P_1, P_2 be any two corresponding points, and if P_1P_2 meet the vanishing line i_1 at I_1 and the axis of perspective at X

$$\frac{OI_1}{OP_1} + \frac{I_1X}{OP_2} = 1.$$

3. Prove (without using the property of the pole of perspective) that the correspondence between two figures in plane perspective is entirely given by the axis of perspective and two pairs of corresponding points. Deduce a construction for the point corresponding to a given point with the above data.

4. Given a pair of corresponding lines and the two vanishing lines of two figures in plane perspective, construct (*a*) the pole of perspective, (*b*) the point corresponding to any given point.

5. Knowing one vanishing line, the axis of perspective, and a pair of corresponding points, construct the pole of perspective.

6. Given the pole and axis of perspective and a pair of corresponding points, construct the two vanishing lines.

17. Drawing of projections. If it be required to draw on paper the projection upon a plane β of any given figure in a plane α from a given vertex V, or, what is the same thing, the section by β of a cone whose vertex is V and base the given figure, the method adopted in practice is to rabat the figure to be drawn upon the plane α about $\alpha\beta$. From the data of the problem $\alpha\beta$ is known. Also drawing through V a plane parallel to β, this plane cuts α in the vanishing line i_1 of the given figure. The pole of perspective is then obtained by rabatting V about i_1 in the same sense that the projection is rabatted about $\alpha\beta$. We have now the pole of perspective, the axis of perspective and one vanishing line. The rabatted projection, which is of course in plane perspective with the original figure, may now be drawn by the rules given in Art. 16. If at any stage the construction becomes awkward, so that lines or points employed in the construction come off the paper, two suitable corresponding points (or lines) may be found and the relevant constructions used.

If the projection be cylindrical, the construction by the vanishing line fails, for by Art. 12, both vanishing lines are then at infinity. Thus to a point $I_1{}^{\infty}$ at infinity corresponds a point $I_2{}^{\infty}$ also at infinity; and $I_1{}^{\infty}$, $I_2{}^{\infty}$ are in general distinct, since the axis of perspective is not here at infinity. Their join $I_1{}^{\infty}I_2{}^{\infty}$ is therefore the line at infinity; and O, which is on this line, is a point at infinity. Its position is then to be found by constructing, in any manner, some one pair of corresponding points A_1, A_2. O^{∞} is then the point at infinity on A_1A_2. The construction for corresponding points which is given first in Art. 16 may then be used, remembering that, where a line is stated to be drawn "through O" in that construction, it should in the present case be drawn parallel to A_1A_2.

If we do this, we find that in Fig. 7 A_1A_2 and P_1P_2 are parallel, so that, if these lines meet x at K and N respectively, $P_2N : P_1N = A_2K : A_1K =$ constant. Hence such a cylindrical projection, when rabatted into the plane of the original figure, is obtainable from it by a stretch (*cf.* Art. 13, Ex. 1).

18. Practical example. A circle of radius 4 units and centre C lies in a horizontal plane α. V is a point 3 units vertically above a point A_1 of the circle. B_1 is a point of the circle 90° distant from A_1. The circle is projected from V on to a plane β passing through a line x in α which bisects CB_1 at right angles. The

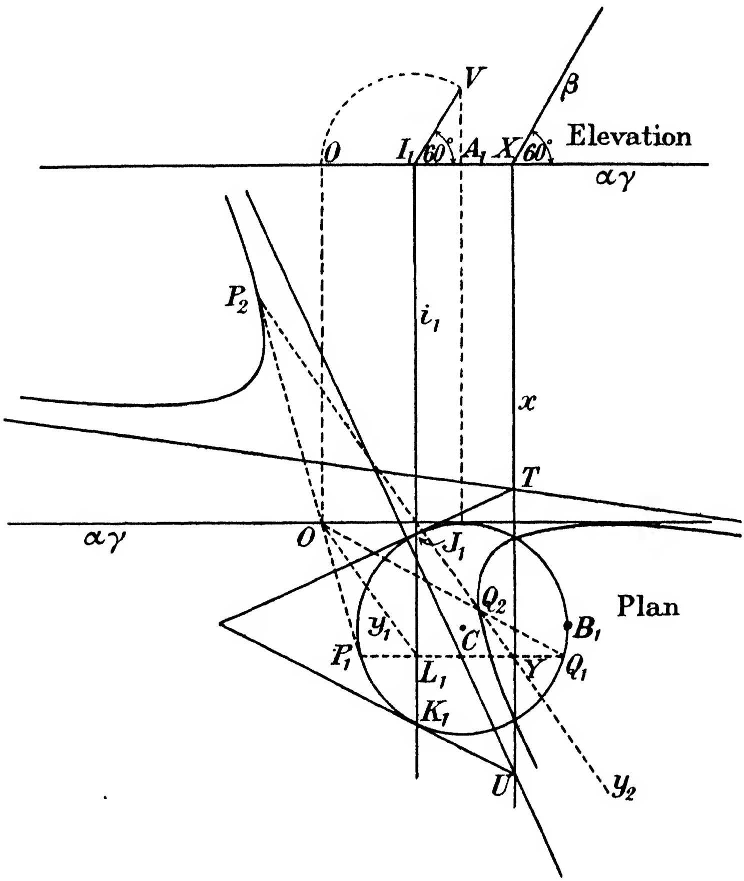


Fig. 8.

plane β is inclined at 60° to the horizontal plane. There are two such planes β. To define β completely we suppose that it is the one whose upper half is further from A_1.

Consider the plane γ which passes through V and is perpendicular to x. We shall need, for the practical construction, two figures (Fig. 8), one in γ which we shall call the elevation figure, and one in α which we shall call the plan figure. In the elevation figure the planes α, β appear as straight lines, viz. the lines in which they cut γ; these are the traces of the planes on γ. Similarly in the plan figure γ appears as its trace on α. It is convenient to place the figures one above the other, the two lines which represent $\alpha\gamma$ in the two figures being parallel, the points which represent the same points being on the same perpendiculars to $\alpha\gamma$.

Mark in the elevation figure the point A_1 and the point X where x meets $\alpha\gamma$. Through X draw a line making 60° with $\alpha\gamma$. This is the trace of β. V is 3 units above A_1 in the elevation figure. Through V draw VI_1 parallel to the trace of β to meet $\alpha\gamma$ at I_1. I_1 is thus a point on the vanishing line of the original figure. Rotate V about I_1 counterclockwise into a position O on $\alpha\gamma$. O is the pole of perspective when the figure in plane β is rabatted about x counterclockwise. Let the original figure and its rabatted projection be denoted by ϕ_1, ϕ_2 respectively. Then in the plan figure x is the axis of perspective, O is the pole of perspective, the parallel i_1 to x through I_1 is the vanishing line of ϕ_1.

To construct the figure corresponding to the circle we have the following method. Let L_1 be a fixed point on i_1. Take a variable point Y on x. Through Y draw a parallel y_2 to OL_1. Join $L_1Y = y_1$, meeting the circle at P_1, Q_1. OP_1, OQ_1 meet y_2 at the points P_2, Q_2 corresponding to P_1, Q_1. By taking a number of parallels y_2 we obtain a number of points on the projection of the circle. This projection is shown by the curve in Fig. 8.

The lines corresponding to the tangents at the points J_1, K_1 where i_1 meets the circle are important. These tangents at infinity or **asymptotes** (*see* later, Art. 34) are immediately constructed by drawing through the points T, U where the tangents to the circle at J_1, K_1 meet x, parallels to OJ_1, OK_1.

19. Problems in projection. It is often useful to be able to construct a projection so that the projected figure shall satisfy certain conditions. We will consider three of these.

I. To project a figure ϕ_1 so that a given line i_1 is projected to infinity. Thus i_1 is to be the vanishing line. Hence, the vertex V being arbitrarily selected, the plane of projection is any plane parallel to Vi_1.

II. To project a figure ϕ_1 so that a given line i_1 is projected to infinity and the angle between two given lines a_1, b_1 is projected into a given angle α.

First solve the problem: to construct a plane perspective relation satisfying the required condition. Let A_1, B_1 be the points where a_1, b_1 respectively meet i_1. On A_1B_1 describe a segment of a circle containing an angle α. The pole of perspective O lies on this segment. Take for O any such point and for axis x any line parallel to i_1. This defines a plane perspective relation satisfying the given conditions. Now rotate O about i_1 through any angle θ into a position V, and at the same time rotate the plane of the

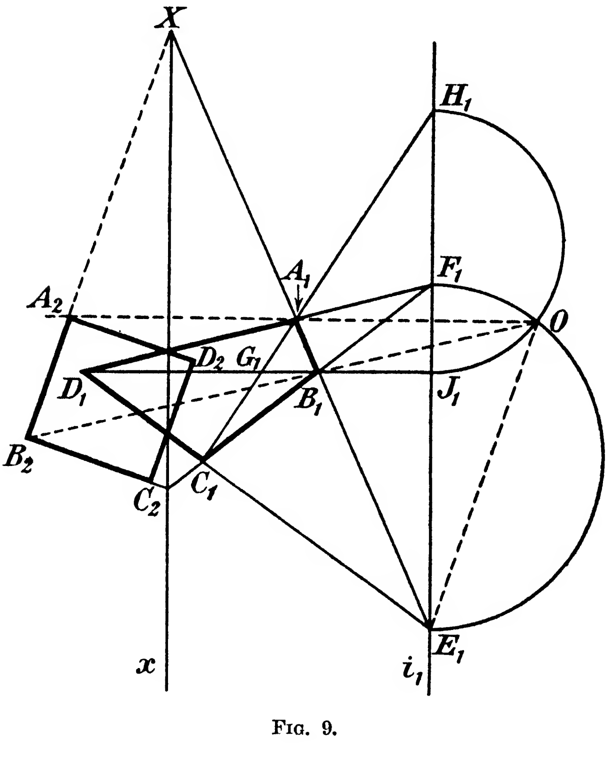


FIG. 9.

original figure about x through the same angle θ into a position β. A projection from V on to β effects what is required.

III. To project a figure ϕ_1 so that a simple quadrilateral $A_1B_1C_1D_1$ (Fig. 9) becomes a square of given size. As in II we will solve the problem first for plane perspective.

Let E_1, F_1 be the intersections of opposite sides (A_1B_1, C_1D_1), (A_1D_1, B_1C_1) respectively; let G_1 be the intersection of the diagonals (A_1C_1, B_1D_1).

Take E_1F_1 as vanishing line i_1; then E_2, F_2 are at infinity, and $A_2B_2C_2D_2$ is a parallelogram.

If the angle at G_2 (the angle between the new diagonals) is a right angle, the parallelogram $A_2B_2C_2D_2$ is a rhombus. If further any one of the angles at A_2, B_2, C_2, D_2 is a right angle, $A_2B_2C_2D_2$ is a square.

Describe on E_1F_1 a semicircle; if O lie on this semicircle the angles at A_1, B_1, C_1, D_1, which stand on E_1F_1, project into right angles.

Similarly if A_1C_1 meet E_1F_1 at H_1 and B_1D_1 meet E_1F_1 at J_1, O lies on a semicircle on J_1H_1. It is therefore the intersection of these two semicircles.

Now the side A_2B_2 must be parallel to OE_1, for E_1 is the vanishing point of A_1B_1. Also A_2 lies on OA_1, B_2 lies on OB_1. Place between OA_1, OB_1, parallel to OE_1, a length A_2B_2 equal to the side of the given square: this will be the line corresponding to A_1B_1. It meets A_1B_1 at a point X on the axis of perspective. Through X draw a parallel to the vanishing line E_1F_1; this is the axis x of perspective.

To obtain the required result by direct projection, rotate O about E_1F_1 through any angle into a position V, and project from V on to a plane through x parallel to VE_1F_1.

EXAMPLES IA

1. Prove that the figures in plane perspective with a given figure, when the vanishing line of that figure and the pole of perspective are given, but the axis of perspective is varied, are similar and similarly situated.

2. Two figures ϕ_1, ϕ_2 are in plane or in space perspective. Lines p_1, q_1 of ϕ_1 are parallel to fixed directions and are such that the angle between them, measured by the rotation in a prescribed sense which brings p_1 into coincidence with q_1, corresponds to a constant angle in ϕ_2. Show that the intersection of p_2 and q_2 describes a circle.

3. Show that, given any two triangles in a plane, a third triangle which is in plane perspective with each of them may be constructed in an infinite number of ways.

4. If a figure ϕ_1 is in plane perspective with ϕ_2 and ϕ_2 in plane perspective with ϕ_3, O_1, O_3 being the poles of perspective and x_1, x_3 the axes of perspective in the two cases, show that O_1O_3, x_1, x_3 form a self-corresponding triangle in ϕ_1, ϕ_3. What happens when O_1, O_3, x_1x_3 are collinear?

5. Given any two triangles in space, prove that a third triangle can always be found which is in space perspective with each of the original two.

6. Prove that a rotation of a figure in its own plane π about a point O of that plane through a given angle θ can be effected by three projections, as follows.

Take a plane α through O perpendicular to the given plane π, project from π on to α with any vertex U. Rotate α, U about the perpendicular to π through O, the angle of rotation being θ. Let this bring α to β and U to V. Project from α upon β with vertex W^∞ in the disection perpendicular to the plane bisecting the dihedral angle θ between α, β. Finally project from β upon π with vertex V.

7. Two sets of four points A_1, B_1, C_1, D_1; A_2, B_2, C_2, D_2, in the same plane, are such that A_1A_2, B_1B_2, C_1C_2, D_1D_2, are concurrent at a point O. Show that, in general, the figures formed by the four points are not in plane perspective, and that the necessary and sufficient condition that they should be so is that three intersections such as (A_1B_1, A_2B_2) (A_1C_1, A_2C_2) (A_1D_1, A_2D_2) are collinear.

8. Prove that another way of expressing the condition that the sets of four points in Ex. 7 shall be in plane perspective is that one pair of corresponding intersections such as (A_1B_1, C_1D_1) and (A_2B_2, C_2D_2) shall be in a line through O, and that, when this condition is satisfied, all points and lines derived from the original sets of four by taking corresponding joins and meets form two figures in plane perspective.

9. If in a plane perspective relation it is given that the pole of perspective and the axis of perspective are at infinity, show that the perspective relation must be equivalent to a translation without rotation in the plane.

10. Show how to project a given line to infinity and at the same time any two given angles into angles of given magnitude. Is this problem capable of solution in all cases?

11. Show how to project a given line to infinity and a given triangle into a triangle congruent with a given triangle.

12. Show how to project a simple quadrilateral into a parallelogram congruent with a given parallelogram.

13. A triangle ABC has its sides AB, AC cut at D and E by a parallel to the base. Show how to construct an equilateral triangle of given side which shall be in plane perspective with ABC, DE being taken as the vanishing line.

14. In Problem III of Art. 19 show that there are two possible positions of O and two possible positions of x and that these may be combined in pairs in four ways, so that there are four perspective relations giving a solution of the problem.

15. Prove that, if A_1, B_1, C_1, D_1 are any four given points (no three of which are collinear) of a plane figure ϕ_1, and A_2, B_2, C_2, D_2 are any four given points (no three of which are collinear) of a plane figure ϕ_2 (not necessarily in the same plane as ϕ_1), then it is, in general, possible to obtain a series of projections which transform $A_1B_1C_1D_1$ into $A_2B_2C_2D_2$.

16. Show that if two figures are similar (but not necessarily similarly situated) the vanishing lines are at infinity.

17. Three coplanar triangles are two by two in perspective and have a common axis of perspective. Show that the poles of perspective are collinear.

18. Three coplanar triangles are two by two in perspective and have a common pole of perspective. Show that the axes of perspective are concurrent.

EXAMPLES Ib

[The axes of co-ordinates are rectangular throughout.]

1. Two figures in plane perspective have $x=0$ for axis of perspective. $A_1=(2, 0)$; $A_2=(2{\cdot}5, 2{\cdot}5)$; $B_1=(3, 0)$; $B_2=(1, 1)$ are pairs of corresponding points. Without using the property of the pole of perspective, construct points corresponding to $P_1=(2, 3)$; $Q_2=(5, -4)$; I_1^∞ at infinity on $y=0$; J_2^∞ at infinity on $x+y=0$. Verify that A_1A_2, B_1B_2, P_1P_2, Q_1Q_2, I_1I_2, J_1J_2 all pass through a point.

2. The pole of perspective being the origin, the axis of perspective the line $x+2=0$ and the vanishing line of the figure ϕ_1 being $x=8$, construct the points of ϕ_2 corresponding to $(-\frac{1}{2}, 4)$, $(-1, -1)$, $(1, -2)$, $(2, 3)$; construct also the points of ϕ_1 corresponding to the same points.

3. Given the pole of perspective $(3, 0)$, the axis of perspective $x=0$ and the pair of corresponding lines a_1 $(y=x)$ and a_2 $(2y=x)$ construct by tangents the curve corresponding to the circle $x^2+y^2=4$ of the figure ϕ_1.

4. $ABCD$ is a square of 3 inches side. E, F, G are points of AD, AB and BC respectively such that $AF=DE=CG=1$ inch; EB and GF meet at O. A', D' are the mid points of OA, OD respectively, and B', C' are the points of trisection (nearest to O) of OB and OC respectively. Construct the pole and axis of perspective which transform A, B, C, D into A', B', C', D', and verify (i) that this axis is parallel to AD and BC, (ii) that the diagonals AC, BD of the square meet the corresponding diagonals $A'C'$, $B'D'$ of the quadrilateral upon the axis.

5. AB is a diameter of a horizontal circle of radius 2 inches. An oblique cone is formed by projecting this circle from a point V vertically above A, the distance VA being 2 inches. Draw the section of this cone by the plane through A at right angles to VB.

6. Two planes α_1, α_2 cut one another at an angle of 60°. On the plane bisecting the angle of 120° between them a vertex V is taken distant 4 inches from their line of intersection.

If a figure in α_1 is projected from V on to α_2 construct the vanishing lines of the figure in α_1 and the rabatted projection. If the axis of perspective be taken for axis of y and the foot of the perpendicular from V upon it as origin and if the positive half of the axis of x be the one nearer to V, find the points in α_2 corresponding to $(2, 0)$, $(5, 0)$, $(3, 4)$ in α_1.

7. A pyramid 80 feet high stands on a square base of side 100 feet, the sides of the base running N. and S., E. and W. Draw the section of this pyramid by a plane at 30° to the horizontal passing through a line running from W.N.W. to E.S.E. through the S.W. corner of the pyramid, the plane rising as one moves N.

8. $ABCD$ is a horizontal square, of side 2 inches; E is a point of BC such that $BE=3.EC$; a point S is taken, outside the square, on the perpendicular from D to AE, so that $DS=\frac{1}{2}AC$. V is the point vertically above S, such that $VS=DS$. Construct the projection of the square $ABCD$ from the point V on to the vertical plane through B parallel to AE.

9. A right circular cone of semi-vertical angle 60° is cut by a plane making an angle of 30° with its axis and cutting that axis at a distance of 3 inches from the vertex. Draw the curve of section.

10. A horizontal square $ABCD$ of 2″ side is projected from a vertex 1·7″ above the corner A. Draw its projections upon the two planes through the diagonal BD inclined at 45° to the plane of the square.

11. A convex quadrilateral $ABCD$ is such that $AB=4''$, $AD=5''$, $CD=2''$, $CB=3''$, $AC=5''$. Find a pole and axis of perspective which will transform $ABCD$ into a square of side $1''$ and draw this square.

12. The axis of x being taken as vanishing line, construct an equilateral triangle of side 2 units which is in plane perspective with the triangle whose vertices are (1, 2), (2·5, 2·5), (3, 1); and construct the pole and axis of perspective for this case.

13. A circular cylinder of radius $2''$ is cut by a plane making an angle of 37° with its axis. Draw the section.

14. A horizontal circle is projected on to a vertical plane through its centre from a point at infinity on a ray inclined at 45° to the vertical and such that the vertical plane through it is inclined at 60° to the plane of projection. Draw the projection.

15. The entrance of a skew tunnel is in the shape of a circular arch; the horizontal projection of the axis of the tunnel makes an angle of 15° with the normal to the plane of the arch and the axis itself slopes upwards at 30°. Draw the section of this tunnel by a horizontal plane.

CHAPTER II

CROSS-RATIO ; PROJECTIVE RANGES AND PENCILS.

20. Cross-ratio. Let A_1, B_1, C_1, D_1 (Fig. 10), be four points on a straight line u_1. Let them be projected from any vertex V into points A_2, B_2, C_2, D_2, upon another straight line u_2.

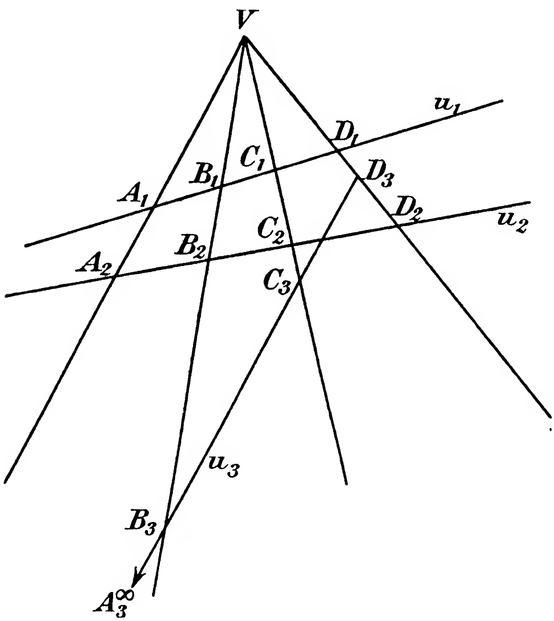


Fig. 10.

We require to find a relation between the mutual distances of the points A_1, B_1, C_1, D_1, which will not be altered by projection.

Consider first the ratio of two segments.

$$\begin{aligned} A_1B_1 : A_1D_1 &= \triangle A_1VB_1 : \triangle A_1VD_1 \\ &= VA_1.VB_1 \sin A_1VB_1 : VA_1.VD_1 \sin A_1VD_1 \\ &= VB_1 \sin A_1VB_1 : VD_1 \sin A_1VD_1. \end{aligned}$$

Note carefully that the above equation holds whatever the relative positions of A_1, B_1, D_1 (the signs of the segments being taken into account as explained in Art. 2), provided we introduce the following conventions as to sign. On each ray through V a positive sense is arbitrarily assigned, which fixes the signs to be attached to the segments VA_1, VB_1, etc. Also a positive sense of rotation round V is arbitrarily selected. The angle A_1VB_1 is then defined as the magnitude (positive or negative) of the rotation in this sense required to bring the positive direction on VA_1 into coincidence with the positive direction on VB_1. This rotation is clearly unique save for the addition or subtraction of a number of complete turns, which does not affect the values of the sines.

In like manner

$$A_2B_2 : A_2D_2 = VB_2 \sin A_1VB_1 : VD_2 \sin A_2VD_2.$$

$$\therefore \quad (A_1B_1 : A_1D_1) \div (A_2B_2 : A_2D_2) = (VB_1 : VD_1) \div (VB_2 : VD_2).$$

The right hand side of the above equation is independent of the points A_1, A_2. It depends only on the bounding rays VB_1B_2, VD_1D_2. We may therefore replace A_1 by C_1, A_2 by C_2, without altering the value of the left hand side. We have then

$$(A_1B_1 : A_1D_1) \div (A_2B_2 : A_2D_2) = (C_1B_1 : C_1D_1) \div (C_2B_2 : C_2D_2)$$

or

$$\frac{A_1B_1.C_1D_1}{A_1D_1.C_1B_1} = \frac{A_2B_2.C_2D_2}{A_2D_2.C_2B_2}.$$

The expression $\dfrac{A_1B_1.C_1D_1}{A_1D_1.C_1B_1}$ is termed the **cross-ratio** or the **anharmonic ratio** of the four points A_1, B_1, C_1, D_1 taken in the given order and is denoted by the symbol $\{A_1B_1C_1D_1\}$. To remember it, note that the numerator is obtained by writing down the four points in the given order and the denominator is obtained from the numerator by interchanging the second and fourth elements. We have, then, from the last written equation the theorem :

The cross-ratio of any four collinear points is unaltered by projection.

We shall often have to deal with a cross-ratio $\{ABCD\}$ when one of the four points, say A, is at infinity. This we shall interpret as the limit of the cross-ratio, when A moves away indefinitely on the straight line.

In this case the factor $\dfrac{AB}{AD} = 1 + \dfrac{DB}{AD}$. Since DB is here constant and AD increases numerically without limit, $\dfrac{DB}{AD}$ tends to zero, and $\dfrac{AB}{AD}$ tends to unity. Hence $\{ABCD\}$ approaches as its limit the other factor $\dfrac{CD}{CB}$, and this will be the cross-ratio $\{A^\infty BCD\}$.

To save needless repetition, a ratio such as $A^\infty B : A^\infty C$ will in future be equated to 1, and a ratio such as $BC : A^\infty D$ to zero.

21. Different cross-ratios of four points. If we take the points A_1, B_1, C_1, D_1 in a different order, we obtain in general a different cross-ratio. Since four letters may be written down in 24 different orders, we should expect 24 different cross-ratios. It will now be shown that only six of these are distinct.

First we shall prove that the cross-ratio of four points is unaltered if any two points be interchanged, provided the remaining two be also interchanged. Since under these circumstances the first point A_1 must necessarily be interchanged with some other, the three cases to be considered are therefore those where A_1 is interchanged with B_1, C_1 and D_1 respectively. We have to prove that

$$\{A_1B_1C_1D_1\} = \{B_1A_1D_1C_1\} = \{C_1D_1A_1B_1\} = \{D_1C_1B_1A_1\},$$

or, writing out the cross-ratios,

$$\frac{A_1B_1.C_1D_1}{A_1D_1.C_1B_1} = \frac{B_1A_1.D_1C_1}{B_1C_1.D_1A_1} = \frac{C_1D_1.A_1B_1}{C_1B_1.A_1D_1} = \frac{D_1C_1.B_1A_1}{D_1A_1.B_1C_1},$$

equalities which are obviously true.

It follows that distinct cross-ratios can be derived only from those permutations in which A_1 stands first. For, if we have any permutation in which A_1 does not stand first, it may be converted into a permutation in which A_1 does stand first by permuting A_1 with the leading element and interchanging the remaining two elements, and this without altering the cross-ratio.

We have then only six distinct cross-ratios, namely those in which A_1 stands first, the remaining three B_1, C_1, D_1 being permuted in all possible ways.

To find the relations among these ratios, project A_1 to infinity, that is, cut the four rays through V by a straight line u_3 (Fig. 10) parallel to VA_1. We have by Art. 20

$$\{A_1B_1C_1D_1\} = \{A_3{}^\infty B_3C_3D_3\} = \frac{C_3D_3}{C_3B_3} = \lambda, \text{ say } \ldots\ldots (1)$$

Interchange even letters D and B. Then

$$\{A_1D_1C_1B_1\} = \frac{C_3B_3}{C_3D_3} = \frac{1}{\lambda} \text{ ....(2)}$$

Interchange middle letters B and C,

$$\{A_1C_1B_1D_1\} = \frac{B_3D_3}{B_3C_3} = \frac{D_3B_3}{C_3B_3} = \frac{D_3C_3 + C_3B_3}{C_3B_3}$$

$$= 1 - \frac{C_3D_3}{C_3B_3} = 1 - \lambda \text{ ....(3)}$$

Interchange in (2) the middle letters,

$$\{A_1C_1D_1B_1\} = 1 - \frac{1}{\lambda} = \frac{\lambda - 1}{\lambda} \text{ ....(4)}$$

Interchange second and fourth letters in (3),

$$\{A_1D_1B_1C_1\} = \frac{1}{1-\lambda} \text{ ....(5)}$$

Interchange second and fourth letters in (4),

$$\{A_1B_1D_1C_1\} = \frac{\lambda}{\lambda - 1} \text{ ....(6)}$$

These give the six distinct cross-ratios of four points.

Examples

1. Show that the six cross-ratios of four points may be expressed in the form $\sin^2\theta$, $\cos^2\theta$, $\operatorname{cosec}^2\theta$, $\sec^2\theta$, $-\tan^2\theta$, $-\cot^2\theta$.

2. If A, B, C, D, are four collinear points, and $\{ACBD\} = m$, $\{ACBE\} = n$, prove that $\{ADCE\} = (n-1)/(m-1)$.

3. If A, B, C, D are four collinear points, prove that

$$BC.AD + CA.BD + AB.CD = 0.$$

Hence express the six cross-ratios of four points in terms of $\{ABCD\}$.

22. Cross-ratio of four rays. From Art. 20 it follows that all transversals, that is, all straight lines which cut a set of four rays or lines through a point, meet the four rays in sets of four points which have the same cross-ratio. This cross-ratio is therefore a property of the set of four rays and is called the cross-ratio of the four rays.

The analytical expression for the cross-ratio of four such rays is easily written down. For

$$\frac{A_1B_1.C_1D_1}{A_1D_1.C_1B_1}=\frac{\triangle A_1VB_1}{\triangle A_1VD_1}\cdot\frac{\triangle C_1VD_1}{\triangle C_1VB_1}$$
$$=\frac{VA_1.VB_1\sin A_1VB_1.VC_1.VD_1\sin C_1VD_1}{VA_1.VD_1\sin A_1VD_1.VC_1.VB_1\sin C_1VB_1}$$
$$=\frac{\sin A_1VB_1.\sin C_1VD_1}{\sin A_1VD_1.\sin C_1VB_1},$$

the signs of the angles following the convention of Art. 20.

Since four concurrent rays project into four concurrent rays and transversals into transversals, it follows from the permanence of cross-ratio of four points in projection that the cross-ratio of four rays is likewise unaltered by projection.

The cross-ratio of four rays *abcd* will be denoted by $\{abcd\}$. If the rays be OA, OB, OC, OD, it will also be denoted by $O\{ABCD\}$.

23. Ranges and pencils. A **range** is a set of points on a straight line. A **flat pencil,** or shortly a **pencil,** is a set of rays lying in a plane and passing through a point which is the **vertex** or **centre** of the pencil.

Ranges and pencils are called one-dimensional elementary geometric forms. The component points or rays are spoken of as the **elements** of the form. The straight line containing the points of a range is termed the **base** of the range. Similarly the vertex of a flat pencil is a base of the pencil, and, so long as we deal with pencils in one plane, this is the only base which need be considered. More generally, however, a flat pencil has *two* bases, its vertex and its plane.

Where only a limited number of elements of a form are considered, the form may be denoted by enclosing the set of elements in round brackets, thus $(ABCD)$ denotes a range consisting of four points A, B, C, D. Similarly $O(ABCD)$ denotes a pencil consisting of the four rays OA, OB, OC, OD. Care should be exercised to use *round* brackets in this connection, so as to avoid confusion with $\{ABCD\}$, which denotes a cross-ratio.

More frequently a form will be denoted by taking a typical element and enclosing it in square brackets. Thus $[P]$ is a range of which the point P, which is then considered variable, is the typical element: $[p]$ is a pencil of which p is the typical ray, or $O[P]$ is a pencil with vertex O, of which OP is the typical ray.

24. Projective ranges and pencils. The elements of two forms may be made to correspond, each to each. When the forms are of the same type, that is, when both are ranges or both pencils, they are said to be projective with one another when the correspondence can be established by means of a finite number of projective operations. It follows from the definition that, if two forms are projective with a third form, they are projective with one another. For the series of projective operations which transform the first form into the third, combined with the series which transforms the second into the third, applied in the reverse order, transforms the first into the second. (*Cf.* Art. 14.)

It follows further from Arts. 20, 22 that projective ranges and pencils are also equi-anharmonic, that is, any four elements of one form have the same cross-ratio as the four corresponding elements of any other form projective with the first.

An important particular case of projective ranges and pencils is when the two ranges are sections of the same pencil by two different transversals, or when the two pencils are obtained by joining up the points of the same range to two different vertices. In the first case the joins of corresponding points of the two ranges pass through a fixed point : in the second case the meets of corresponding rays of the two pencils lie on a fixed line. Two such ranges and pencils are said to be **perspective :** they are clearly particular cases of figures in plane or in space perspective, and are therefore projective.

If two ranges be perspective the point where their bases intersect is self-corresponding, and if two coplanar pencils be perspective the ray joining the two vertices is self-corresponding.

Similar ranges are corresponding ranges in which corresponding segments are proportional.

Equal ranges or pencils are ranges and pencils which can be superposed so that corresponding elements coincide.

Equal pencils in one plane are said to be **directly,** or **oppositely,** equal according as they can, or cannot, be superposed without being turned over.

Since it has been shown (Art. 15) that congruent and similar figures are particular cases of projective figures, it follows that similar ranges are projective and also that equal ranges and equal pencils are projective.

In two similar ranges the points at infinity correspond. For since $A_1B_1 : A_2B_2 =$ a finite ratio λ, if A_1B_1 is infinite, so must

A_2B_2 be infinite. Hence if A_1, A_2 be accessible points and B_1 a point at infinity, B_2 is at infinity.

Conversely projective ranges in which the points at infinity correspond are similar. Let $A_1B_1C_1I_1^\infty$, $A_2B_2C_2I_2^\infty$ be corresponding groups of four points of two such ranges, then, since the cross-ratio is unaltered,

$$\frac{A_1B_1.C_1I_1^\infty}{A_1I_1^\infty.C_1B_1}=\frac{A_2B_2.C_2I_2^\infty}{A_2I_2^\infty.C_2B_2},$$

and remembering (Art. 20) that $\dfrac{C_1I_1^\infty}{A_1I_1^\infty}=1$ and $\dfrac{C_2I_2^\infty}{A_2I_2^\infty}=1$, we have

$$A_1B_1 : C_1B_1 = A_2B_2 : C_2B_2,$$

and therefore the ranges are similar.

Projective ranges may be **collinear,** that is, a projective correspondence can be established between points of the same line; in this case a particular point of the base has a different significance, according as we consider it to belong to one range or to the other. Similarly projective pencils may be **concentric** (that is, they have a common vertex) as well as coplanar. They then consist of the same rays, but each ray has a different significance, according as we assign it to one pencil or to the other. Two such ranges or two such pencils will be termed **cobasal** ranges or pencils.

Sections of two projective pencils $[p_1]$, $[p_2]$ by transversals v, w are projective.

For in the set of projections which transform $[p_1]$ to $[p_2]$ let a line u_1 not belonging to $[p_1]$ transform into u_2.

The range $u_1[p_1]$ is projective with $u_2[p_2]$.

But $v[p_1]$ is perspective and $\therefore$ projective with $u_1[p_1]$, and $w[p_2]$ is perspective and $\therefore$ projective with $u_2[p_2]$.

Hence $v[p_1]$ is projective with $w[p_2]$.

Similarly if $[P_1]$, $[P_2]$ be two projective ranges, O, S any two vertices, the pencils $O[P_1]$, $S[P_2]$ are projective.

For in the set of projections which transform $[P_1]$ to $[P_2]$ let a point U_1 not belonging to $[P_1]$ transform into U_2.

The pencil $U_1[P_1]$ is projective with the pencil $U_2[P_2]$.

But $O[P_1]$ is perspective and $\therefore$ projective with $U_1[P_1]$, and $S[P_2]$ is perspective and $\therefore$ projective with $U_2[P_2]$.

Hence $O[P_1]$ is projective with $S[P_2]$.

To abridge proofs the words "is projective with" will in future (except in enunciations) be denoted by the symbol $\barwedge$. Thus

$[P_1] \barwedge [P_2]$ is to be read: the range described by P_1 is projective with the range described by P_2.

Two unlike corresponding forms will be called **incident** if each element of one is incident with the corresponding element of the other, provided the correspondence is unique. Thus a range and a pencil are incident if each ray of the latter passes through the corresponding point of the former. This relation is included by Reye under the name perspective, but, since it does not come under our definition of perspective figures, this would lead to some confusion. The use of the term perspective will therefore be restricted as explained earlier in the present article.

25. Two cobasal projective forms are identical if they have three elements self-corresponding.

Consider two ranges. Let A, B, C be the self-corresponding points, P_1, P_2 any two corresponding points.

Then
$$\{ABCP_1\} = \{ABCP_2\},$$
$$\therefore \frac{AB.CP_1}{AP_1.CB} = \frac{AB.CP_2}{AP_2.CB},$$
$$\frac{CP_1}{AP_1} = \frac{CP_2}{AP_2},$$
$$\therefore \frac{CA+AP_1}{AP_1} = \frac{CA+AP_2}{AP_2},$$
$$\therefore CA.AP_2 = CA.AP_1.$$

CA is not zero, since by hypothesis the points A, B, C are distinct, $\therefore AP_1 = AP_2$ or P_1, P_2 are coincident. Hence every point is self-corresponding and the ranges are identical.

Consider now two concentric pencils. They determine on any line two collinear projective ranges. If three rays of the pencils are self-corresponding, three points of the ranges are self-corresponding. Therefore every point of the ranges is self-corresponding and in consequence every ray of the pencils is self-corresponding.

It follows that two distinct cobasal projective forms cannot have more than two self-corresponding elements.

26. Construction of projective ranges and pencils from corresponding triads.

I. Ranges. We need only consider the problem of establishing a projective correspondence between two ranges on different lines but in the same plane; for, if the ranges are in different planes, or in the same straight line, we may first of all project one of them into a range lying in the same plane as the other range, but in a different straight line.

Let u_1, u_2 (Fig. 11 (*a*)) be the bases of the two ranges $[P_1]$, $[P_2]$. Let A_1, B_1, C_1 be three given points of $[P_1]$ corresponding to A_2, B_2, C_2 of $[P_2]$. On A_1A_2 take arbitrarily two vertices X, Y. Join XB_1, meeting YB_2 at B_3, XC_1 meeting XC_2 at C_3. Denote B_3C_3 by u_3, and let u_3 meet A_1A_2 at A_3.

Then A_3, B_3, C_3 are in perspective with A_1, B_1, C_1 from X, and with A_2, B_2, C_2 from Y.

From X project $[P_1]$ into $[P_3]$ on u_3, and from Y project $[P_3]$

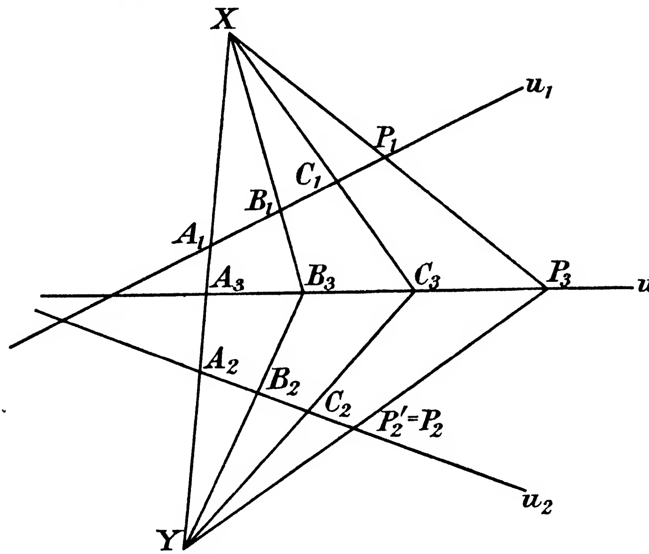


FIG. 11 (*a*).

into $[P_2']$ on u_2. Then $[P_2'] \barwedge [P_1]$; but $[P_1]$ is given projective with $[P_2]$, hence $[P_2'] \barwedge [P_2]$.

These last are cobasal ranges. But clearly A_2', B_2', C_2' are, by the construction described, identical with A_2, B_2, C_2. Hence $[P_2]$, $[P_2']$ have three self-corresponding points and are identical by Art. 25. Thus the projections from X and Y in succession enable us to derive the range $[P_2]$ from the range $[P_1]$.

II. PENCILS. First of all, if the two pencils are not already coplanar and non-concentric, project one of them into a pencil coplanar and non-concentric with the other. We shall then consider the two pencils to be of this type.

Let U_1, U_2 (Fig. 11 (*b*)) be the vertices of two such pencils $[p_1]$, $[p_2]$. Let a_1, b_1, c_1 be three given rays of $[p_1]$, corresponding to a_2, b_2, c_2 of $[p_2]$. Through a_1a_2 (denoted by A) draw two arbitrary rays, x, meeting b_1, c_1 at B_1, C_1, and y, meeting b_2, c_2 at B_2, C_2. Let B_1B_2, C_1C_2 meet at U_3. Let U_3A, U_3B_1, U_3C_1 be a_3, b_3, c_3.

Then the rays a_3, b_3, c_3 are perspective with a_1, b_1, c_1, and also with a_2, b_2, c_2.

Let x meet the pencil $[p_1]$ in the range $[P_1]$. Join the points of this range to U_3, forming the pencil $[p_3]$. Then $[p_3]$ is perspective and therefore projective with $[p_1]$. Let y meet $[p_3]$ in the range

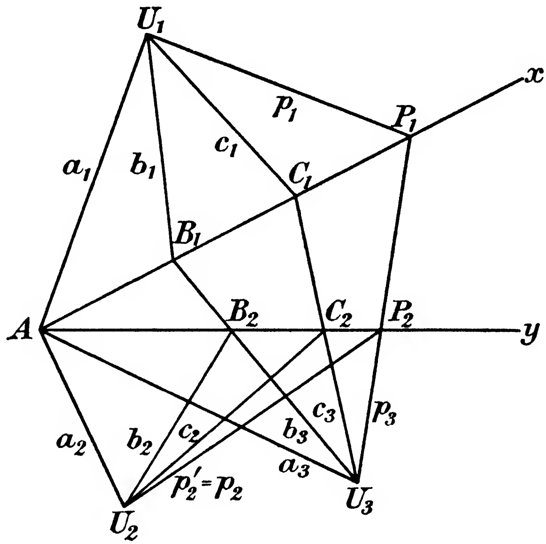


FIG. 11 (*b*).

$[P_2]$. Join the points of $[P_2]$ to U_2, forming the pencil $[p_2']$. Then $[p_2'] \barwedge [p_3] \barwedge [p_1]$, and $[p_1]$ is given projective with $[p_2]$. Thus $[p_2'] \barwedge [p_2]$. But it is clear that a_2', b_2', c_2' are identical with a_2, b_2, c_2. Hence, by Art. 25 $[p_2]$ and $[p_2']$ are identical, and the given construction enables us to derive $[p_2]$ from $[p_1]$.

It follows from the above constructions:

(*a*) That the relation between two projective forms is entirely determined as soon as three corresponding pairs of elements are given.

(*b*) That a projective relation between two like forms can always be established in which three arbitrary elements of one shall correspond to three arbitrary elements of the other, which is sometimes expressed by saying that groups of three elements are always projective.

(*c*) That a projective relation between two like forms can always be established in which any four elements of the one correspond to four elements of the other having the same cross-ratio.

For let A_1, B_1, C_1, D_1 ; A_2, B_2, C_2, D_2 be two sets of four points lying in straight lines u_1, u_2 and such that $\{A_1B_1C_1D_1\} = \{A_2B_2C_2D_2\}$. Let the projective relation which transforms A_1, B_1, C_1 into A_2, B_2, C_2 transform D_1 into D_2'. Then

$$\{A_1B_1C_1D_1\} = \{A_2B_2C_2D_2'\}$$

Therefore $\{A_2B_2C_2D_2\} = \{A_2B_2C_2D_2'\}$, whence it follows as in Art. 25 that $D_2 = D_2'$. A corresponding proof holds for pencils.

The above constructions fail if either A_1 or $A_2 = u_1u_2$, but is not self-corresponding ; or if either a_1 or $a_2 = U_1U_2$, but is not self-corresponding. It is to be noted, however, that this cannot possibly occur for all three pairs of corresponding elements, for if $A_1 = u_1u_2$, B_2 may also be u_1u_2, but then neither C_1 nor C_2 can be u_1u_2. The pair of elements which do not include u_1u_2 (or U_1U_2) can always be taken as A_1, A_2 (or a_1, a_2) in the above constructions, which are then always possible.

If, however, both A_1 and A_2 coincide with u_1u_2 in Fig. 11 (*a*), or both a_1 and a_2 coincide with U_1U_2 in Fig. 11 (*b*), the two given forms have one element self-corresponding. The line XY then passes through u_1u_2, but is otherwise indeterminate ; or the point xy lies on U_1U_2, but is otherwise indeterminate.

A simpler construction can then be given. For let B_1B_2 meet C_1C_2 at U, then AB_1C_1 and AB_2C_2 are perspective from U. Similarly if the join of b_1b_2, c_1c_2 be u, ab_1c_1, ab_2c_2 are perspective, corresponding rays meeting on u. The two given ranges or pencils are then perspective and we have the important result :

If two projective ranges or flat pencils, which are coplanar, but not cobasal, have a self-corresponding element, they are perspective.

Examples

1. Give a geometrical construction connecting the points of two projective ranges when the vanishing points of the ranges and a pair of corresponding points are given.

2. Two collinear projective ranges are given by two corresponding triads $A_1B_1C_1$, $A_2B_2C_2$. Give completely a geometrical construction to find the point P_2 of the second range corresponding to a given point P_1 of the first.

3. Two concentric projective pencils are given by two corresponding triads $a_1b_1c_1$, $a_2b_2c_2$. Give completely a geometrical construction to find the ray p_2 of the second pencil corresponding to a given ray p_1 of the first.

4. Two similar coplanar ranges have a self-corresponding point. Show that the lines joining their corresponding points are all parallel.

5. If two similar ranges lie on parallel lines, the joins of corresponding points pass through a fixed point.

27. Harmonic forms. Since any three collinear points may be projected into any three other points, three points A, B, C on a

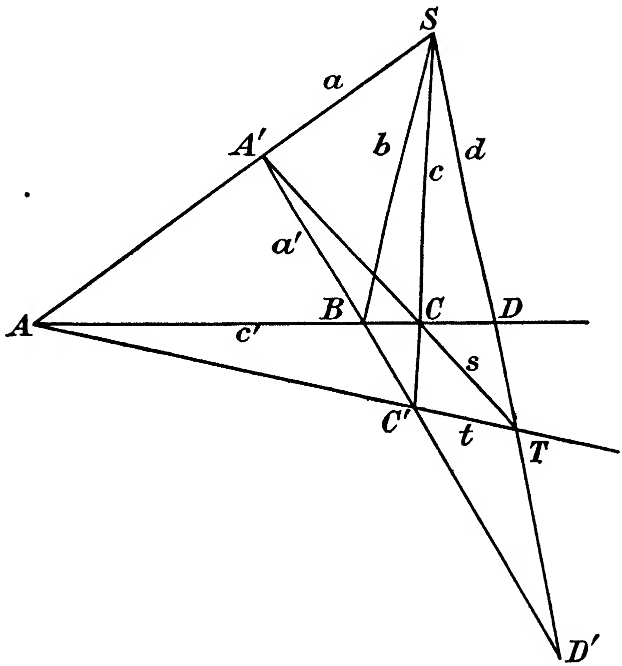


FIG. 12.

line c' (Fig. 12) may be projected into the same points with two of them, say A and C, interchanged.

To effect this, draw any line a' through B and from any vertex S in the plane $a'c'$ project A, B, C upon a' as A', B, C'. Let $(A'C, AC') = T$. Then if we project A', B, C' from T upon c', they project into C, B, A. The double operation has therefore interchanged A and C.

The two triads ABC, CBA define two projective collinear ranges on c'. These two ranges have already a self-corresponding point B. They have therefore at most one other point D which corresponds to itself.

This point D is the point where ST meets c'. For if ST meet a' at D', D projects from S on a' into D' and D' projects back from T on c' into D.

Hence $$\{ABCD\} = \{CBAD\}$$

and D is the only point satisfying this condition.

When four points are such that they are projective with themselves, two of them being interchanged, they are said to be **harmonic,** or to form a **harmonic range,** and the two which are interchanged are said to be **harmonically conjugate** with regard to the other two.

By Art. 21, interchanging both A and C, B and D

$$\{CBAD\} = \{ADCB\}.$$

Hence $$\{ADCB\} = \{ABCD\}.$$

It follows from (*c*) of Art. 26 that if A, B, C, D can be projected into C, B, A, D, they can be projected into A, D, C, B. So that if A, C are conjugate with regard to B, D, so are B, D with regard to A, C.

If we join the points of a harmonic range to a point outside the range, we obtain a pencil of four rays possessing the same property, namely that it is projective with itself, two rays being interchanged. The interchangeable rays are said to be harmonically conjugate with regard to the other two, and the pencil is termed a **harmonic pencil.**

28. Cross-ratio of four harmonic elements. Let λ be the cross-ratio of four harmonic elements, say four points A, B, C, D of a range.

If $\{ABCD\} = \lambda$, then by Art. 21 $\{ADCB\} = \dfrac{1}{\lambda} = \{CBAD\}$.

Hence $$\lambda = \frac{1}{\lambda}, \text{ or } \lambda = \pm 1.$$

Now if λ were $+1$, then, by (3) of Art. 21,

$$\{ACBD\} = 0,$$

therefore $$AC.BD = 0.$$

That is, either C and A, or B and D coincide. But this is not the case, by hypothesis. Hence the cross-ratio of four harmonic elements, in which conjugate elements are not coincident, is -1.

The cross-ratio of a harmonic pencil is also -1, since such a pencil stands on a harmonic range.

It follows at once that every transversal cuts a harmonic pencil in a harmonic range and also that four harmonic elements are necessarily projective with four other harmonic elements, since the two sets have the same cross-ratio.

The relation $\{ABCD\} = -1$ can be put into two other different forms, which are of great importance.

We have $$AB.CD + AD.CB = 0,$$
i.e. $$AB(CA + AD) + AD(CA + AB) = 0;$$
or $$AB.AC + AD.AC = 2.AB.AD,$$
and dividing by $AB.AC.AD$
$$\frac{1}{AB} + \frac{1}{AD} = \frac{2}{AC}.$$

AC is therefore a harmonic mean between AB and AD. Similarly it is also a harmonic mean between BC and DC.

To get the other form, let O be the point midway between two conjugates, say A and C. Substituting into the relation
$$AB.CD + AD.CB = 0,$$
we have
$$(AO + OB)(CO + OD) + (AO + OD)(CO + OB) = 0,$$
$$2.AO.CO + (OB + OD)(AO + CO) + 2.OB.OD = 0.$$
But $$AO = OC = -CO, \quad \therefore \; AO + CO = 0,$$
$$\therefore \; OB.OD = -AO.CO = OA^2.$$

When $\{ABCD\} = -1$, $AB : AD = -CB : CD$, or the points A and C divide BD internally and externally in the same ratio. Hence by Euclid VI. 3 the two bisectors of the angles formed by a pair of straight lines are harmonically conjugate with regard to the two given lines.

Conversely if, in a harmonic pencil, one pair of conjugate lines are at right angles, they bisect the angles formed by the other pair. For let a, c be at right angles. Then if b, d be not equally inclined to a, c let b, d' be equally inclined to a, c: then $\{a, b, c, d'\} = -1 = \{a, b, c, d\}$, $\therefore d = d'$, that is b, d are equally inclined to a, c.

If one of the points of a harmonic range be at infinity its conjugate is midway between the other two. For, let A^∞ be this point, then
$$\frac{A^\infty B.CD}{A^\infty D.CB} = -1, \text{ or } \frac{CD}{CB} = -1,$$
that is $BC = CD$ or C bisects BD.

If $B=D$, or $C=A$, the definition of Art. 27 apparently leads to an indeterminate result. Let us agree that the equation

$$AB.CD+AD.CB=0$$

shall hold in all cases. If we now put $D=B$, we have

$$2AB.CB=0.$$

Hence either $AB=0$ or $CB=0$, that is, either A or C coincides with B and D. That the same result holds for pencils is easily seen on cutting by a transversal.

Examples

1. Find all the cross-ratios of four harmonic points.

2. Prove that the necessary and sufficient condition that four collinear points A, B, C, D can be paired so as to form a harmonic range is that $\{ABCD\}$ has one of the values $(-1, 2, \frac{1}{2})$.

3. Show how to draw through a given point a line which cuts the sides of a given triangle at three points which, taken in a prescribed order with the given point, form a harmonic range.

4. If A, B are harmonically conjugate with regard to C, D, and O is the middle point of AB, prove that

$$\text{(i) } \frac{OD}{OB}=\frac{BD}{CB}=\frac{AD}{AC}=\frac{OB}{OC};$$

$$\text{(ii) } AC.BD=CD.OB.$$

29. Harmonic properties of the complete quadrilateral and quadrangle. A complete quadrilateral is the figure formed by four straight lines a, b, c, d, called its sides. It has six vertices ab, ac, ad, bc, bd, cd formed by taking meets of sides in pairs. The three pairs of vertices ab, cd; ac, bd; ad, bc such that the two in each pair do not lie on a common side are termed pairs of opposite vertices; the three lines joining them are called the **diagonals** of the quadrilateral. The triangle formed by them is the **diagonal triangle** of the quadrilateral.

A complete quadrangle is the figure formed by four points A, B, C, D called its vertices. It has six sides AB, AC, AD, BC, BD, CD formed by taking joins of vertices in pairs. The three pairs of sides AB, CD; AC, BD; AD, BC such that the two in each pair do not pass through a common vertex are termed pairs of opposite sides. Their three meets are called the **diagonal points** of the quadrangle. The triangle formed by them is the diagonal triangle.

The harmonic properties of the complete quadrilateral and quadrangle are as follows:

I. The two vertices of a complete quadrilateral on any diagonal are harmonically conjugate with regard to the two vertices of the diagonal triangle on that diagonal.

II. The two sides of a complete quadrangle through a diagonal point are harmonically conjugate with regard to the two sides of the diagonal triangle through that diagonal point.

To prove these results, refer to Fig. 12. Here AA', $A'C$, CC', $C'A$ are the four sides of a complete quadrilateral, of which $A'C'$, AC, ST are the three diagonals. The diagonal AC is divided harmonically at B and D (Art. 27). But B and D are the points where AC is met by the other two diagonals. The result for the other diagonals follows by symmetry.

Again A, C', C, A' are the four vertices of a complete quadrangle, of which S, B, T are the three diagonal points. The two sides through S, SA and SC, are harmonically conjugate with regard to SB and SD (since A, C are harmonically conjugate with regard to B, D). But SB, SD are the two sides of the diagonal triangle through S.

From the above properties we obtain the following constructions for the element harmonically conjugate to a given element with regard to two other given elements.

I. Through the point B, to which a conjugate is required with regard to A and C, draw any line and on it take any two points A', C' (Fig. 12). Join AA', CC' meeting at S, AC', $A'C$ meeting at T. TS meets the original line in the point D required.

II. On the ray $SB=b$, to which a conjugate is required with regard to $SA=a$, $SC=c$, take any point B, and through it draw any two lines a', c'. Let $s=$join of aa', cc', $t=$join of ac', $a'c$. The join of ts $(=T)$ to the vertex S gives the ray d required.

In the above cases it is often said that D is a fourth harmonic to A, B, C and d a fourth harmonic to a, b, c, respectively.

Examples

1. If EFG be the diagonal triangle of a complete quadrangle $ABCD$ and the sides of EFG also meet the sides of the quadrangle in six other points I, J, K, L, M, N, show that I, J, K, L, M, N are the six vertices of a complete quadrilateral having EFG for its diagonal triangle.

2. If efg be the diagonal triangle of a complete quadrilateral $abcd$ and the vertices of efg be also joined to the vertices of the quadrilateral by six other lines i, j, k, l, m, n, show that i, j, k, l, m, n are the six sides of a complete quadrangle having efg for its diagonal triangle.

3. Show that the centres of the inscribed and escribed circles of any given triangle form a complete quadrangle, of which the triangle is the diagonal triangle.

4. ABC is a triangle. On BC two points U, V are taken, harmonically conjugate with regard to B, C. If P is any point of AB, and VP, UP meet AC at R, S, prove that VR, US meet on AB at a point Q harmonically conjugate to P with regard to A and B.

30. Cross-axis and cross-centre of coplanar projective ranges and pencils. If in construction I. of Art. 26 X be taken at A_2 and Y at A_1 we obtain the case shown by Fig. 13 (a).

Consider the points which correspond to the point of intersection of u_1, u_2. Let this point considered as a point of u_1 be called U_1, and considered as a point of u_2 be called V_2.

A_2U_1 meets u_3 at U_3; and A_2U_1 is itself u_2. Therefore $U_3 = u_2u_3$.

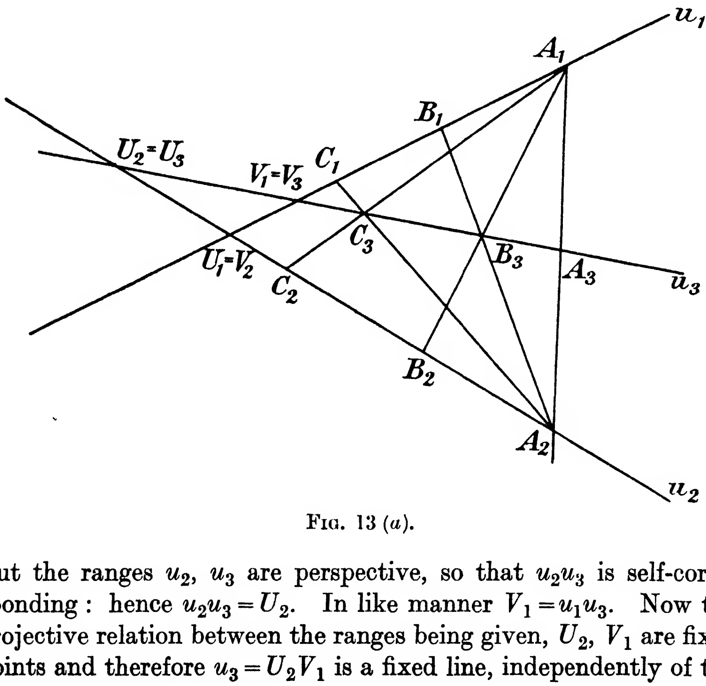


FIG. 13 (a).

But the ranges u_2, u_3 are perspective, so that u_2u_3 is self-corresponding: hence $u_2u_3 = U_2$. In like manner $V_1 = u_1u_3$. Now the projective relation between the ranges being given, U_2, V_1 are fixed points and therefore $u_3 = U_2V_1$ is a fixed line, independently of the choice of $A_1B_1C_1$, $A_2B_2C_2$, which may be any corresponding triads whatever of the given ranges.

It follows that if A_1A_2, B_1B_2 be any two pairs of corresponding points of two projective ranges the meet of cross-joins (A_1B_2, A_2B_1) lies on a fixed straight line. This straight line may be termed the **cross-axis** of the two projective ranges.

Similarly if in construction II. of Art. 26 x be taken coincident with a_2 and y with a_1, we obtain the case shown in Fig. 13 (b). If we now consider U_1U_2 and treat it as a ray u_1 of the pencil $[p_1]$ it meets a_2 at U_2, $\therefore$ $U_3U_2 = u_3$. But the pencils $[p_3]$, $[p_2]$ being perspective, U_3U_2 is self-corresponding, hence $U_3U_2 = u_2$. Similarly if $U_1U_2 = v_2$, $U_1U_3 = v_1$. Hence U_3 is the intersection of the two rays corresponding to U_1U_2; U_3 is therefore a fixed point, independently of the choice of the triads $a_1b_1c_1$, $a_2b_2c_2$. Hence if a_1a_2, b_1b_2 be any two pairs of corresponding rays of two projective pencils the join of cross-meets (a_1b_2, a_2b_1) passes through a fixed

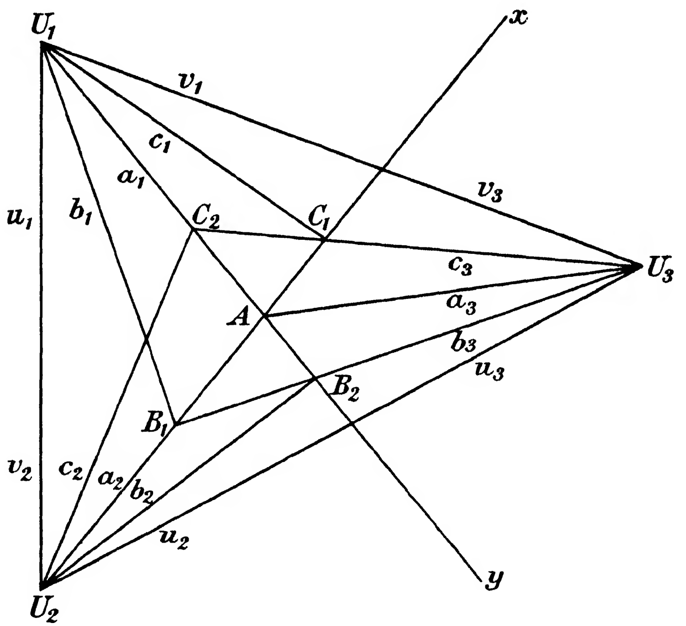


FIG. 13 (*b*).

point. This fixed point will be termed the **cross-centre** of the two projective pencils.

If the ranges (or pencils) in the above theorems be perspective the reasoning employed fails, for then u_1u_2 (Fig. 13 (*a*)) and U_1U_2 (Fig. 13 (*b*)) are self-corresponding. Therefore U_2, V_1 (Fig. 13 (*a*)) and u_2, v_1 (Fig. 13 (*b*)) are coincident, and all we have proved is that u_3 passes through *one* fixed point, viz. the intersection of the ranges, and that U_3 lies on *one* fixed line, viz. the join of the vertices of the pencils.

In the case of perspective ranges and pencils, however, a direct proof of the existence of cross-axis and cross-centre is easily given as follows :

I. FOR RANGES. Let O be the pole of perspective, S the intersection of the ranges, A_1A_2, B_1B_2 two corresponding pairs. Then $A_1A_2B_2B_1$ are vertices of a complete quadrangle of which O, S, (A_1B_2, A_2B_1) are diagonal points. Hence, by the harmonic property of the complete quadrangle, SO and the line joining S to (A_1B_2, A_2B_1) are harmonically conjugate with regard to the bases of the two ranges. But SO and these bases are fixed lines. Hence the line joining S to (A_1B_2, A_2B_1) is a fixed line. Therefore (A_1B_2, A_2B_1) lies on a fixed line, which is the cross-axis.

II. FOR PENCILS. Let x be the axis of perspective, s the join of the vertices, a_1a_2, b_1b_2 two corresponding pairs. Then $a_1a_2b_2b_1$ are sides of a complete quadrilateral of which (a_1b_1, a_2b_2), (a_1a_2, b_1b_2), (a_1b_2, a_2b_1), *i.e.* s, x and (a_1b_2, a_2b_1), are the diagonals. Therefore U_1U_2 is harmonically divided by x and (a_1b_2, a_2b_1). But x meets U_1U_2 at a fixed point, and U_1, U_2 are themselves fixed. Hence the fourth harmonic is also fixed so that (a_1b_2, a_2b_1) passes through a fixed point on s. This is the cross-centre.

We will close the present chapter with the following two theorems on the triangle, which are of importance.

31. The ratio of segments round a triangle. The idea of cross-ratio, as introduced in Art. 20, may be generalised as follows :

If $A_1B_1C_1$, $A_2B_2C_2$ be two triangles in space perspective from a vertex V and any points of section P_1, Q_1, R_1 be taken arbitrarily upon the sides B_1C_1, C_1A_1, A_1B_1 of the first triangle respectively, these will project into points P_2, Q_2, R_2 on the sides B_2C_2, C_2A_2, A_2B_2 of the second triangle.

As in Art. 20 we can show that

$$(B_1P_1 : P_1C_1) \div (B_2P_2 : P_2C_2) = (VB_1 : VC_1) \div (VB_2 : VC_2),$$

and similarly

$$(C_1Q_1 : Q_1A_1) \div (C_2Q_2 : Q_2A_2) = (VC_1 : VA_1) \div (VC_2 : VA_2),$$
$$(A_1R_1 : R_1B_1) \div (A_2R_2 : R_2B_2) = (VA_1 : VB_1) \div (VA_2 : VB_2).$$

Multiply these three sets of equal ratios together and we have

$$\left(\frac{B_1P_1}{P_1C_1} \cdot \frac{C_1Q_1}{Q_1A_1} \cdot \frac{A_1R_1}{R_1B_1}\right) \div \left(\frac{B_2P_2}{P_2C_2} \cdot \frac{C_2Q_2}{Q_2A_2} \cdot \frac{A_2R_2}{R_2B_2}\right) = 1,$$

that is the ratio $\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB}$ of the segments of the sides of a triangle taken in order is unaltered by projection. We may refer to it for brevity as the triangle ratio.

Note carefully that in the above the segments have to be taken with proper sign. The positive sense on each side of the triangle may be arbitrarily selected. It is usual to take it so that, if we go round the triangle keeping the area on our left, we are moving in the positive sense throughout.

Example

Through the vertices A, B, C of a triangle three lines AP, BQ, CR are drawn. Show that the continued product

$$\frac{\sin BAP}{\sin CAP} \cdot \frac{\sin CBQ}{\sin ABQ} \cdot \frac{\sin ACR}{\sin BCR}$$

is unaltered by projection.

32. Ceva's and Menelaus' Theorems. If, in the above, we choose the vertex V and the plane $A_2B_2C_2$ so that the line joining

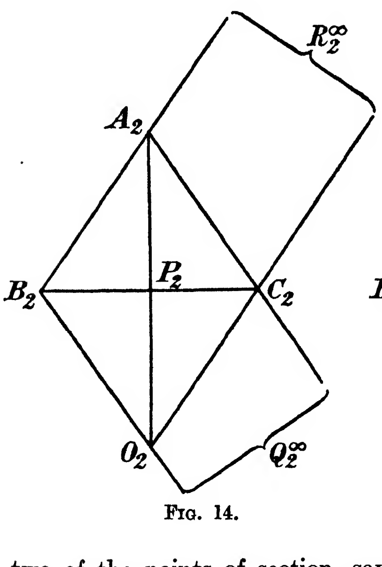


Fig. 14.

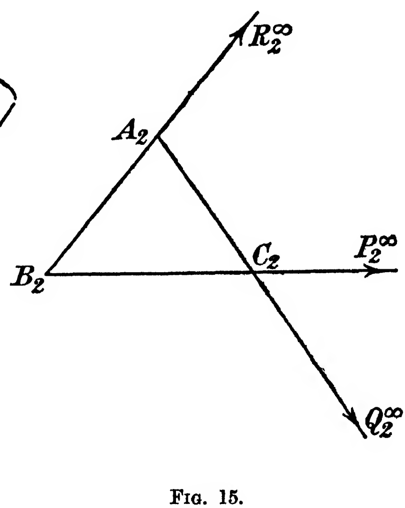


Fig. 15.

two of the points of section, say Q_1, R_1, is projected to infinity, then

$$C_2Q_2{}^\infty : Q_2{}^\infty A_2 = -1 \text{ and } A_2R_2{}^\infty : R_2{}^\infty B_2 = -1$$

and the triangle ratio reduces to $B_2P_2 : P_2C_2$.

Now, if A_1P_1, B_1Q_1, C_1R_1 meet at a point O_1, then O_2 is the intersection of $C_2R_2{}^\infty$ and $B_2Q_2{}^\infty$ (Fig. 14). $A_2C_2O_2B_2$ is then a

5

parallelogram. Its diagonals bisect each other and P_2 is the middle point of B_2C_2. Therefore $B_2P_2 : P_2C_2 = +1$ and

$$\frac{B_1P_1}{P_1C_1} \cdot \frac{C_1Q_1}{Q_1A_1} \cdot \frac{A_1R_1}{R_1B_1} = 1.$$

This is known as Ceva's Theorem.

If, on the other hand, P_1, Q_1, R_1 are collinear, that is, if they are the three points at which any straight line meets the sides of the triangle $A_1B_1C_1$, then in the projected figure (Fig. 15), P_2 is at infinity on BC and $B_2P_2^{\infty} : P_2^{\infty}C_2 = -1$, so that

$$\frac{B_1P_1}{P_1C_1} \cdot \frac{C_1Q_1}{Q_1A_1} \cdot \frac{A_1R_1}{R_1B_1} = -1.$$

This is known as the theorem of Menelaus.

The theorems converse to those of Ceva and Menelaus are easily proved and are left as an exercise for the student.

Examples

1. Prove that the three medians of a triangle meet at a point.

2. Prove that the three perpendiculars from the vertices of a triangle on the opposite sides meet at a point.

3. Prove that the three symmedians of a triangle meet at a point. [A symmedian AD' makes with the sides AB, AC angles equal to those which the median AD makes with AC, AB respectively.]

4. Lines through the vertices of a triangle ABC, equally inclined to the bisectors of the angles, meet the opposite sides at D, D'; E, E'; F, F', respectively. If AD, BE, CF are concurrent, prove that AD', BE', CF' are concurrent.

5. A line cuts the sides BC, CA, AB of a triangle ABC at L, M, N; L', M', N' are the harmonic conjugates of L, M, N with regard to (B, C), (C, A), (A, B) respectively. Show that L', M', N' are collinear.

6. Given three unequal circles, whose centres A, B, C are not in line, show that their six centres of similitude lie in threes on four straight lines, which form a complete quadrilateral of which ABC is the diagonal triangle.

7. The vertices of a triangle are joined to the points of contact of the opposite sides with one of the escribed circles. Show that the lines thus formed are concurrent.

8. Pairs of points P, P'; Q, Q'; R, R' are taken on the sides BC, CA, AB of a triangle and equidistant from their midpoints. Show that if AP, BQ, CR are concurrent, then so also are AP', BQ', CR'.

EXAMPLES IIA

1. Prove that if I_1, J_2 be the vanishing points of two projective ranges, P_1, P_2 any pair of corresponding points, then

$$I_1P_1 . J_2P_2 = \text{constant}.$$

2. Prove that, in two projective ranges, the order of three points A_1, B_1, C_1 is different from, or the same as, the order of the three corresponding points

A_2, B_2, C_2 according as they do, or do not, include the vanishing point between them. Show also that if I_1 is intermediate between two of A_1, B_1, C_1, then J_2 is intermediate between two of A_2, B_2, C_2.

3. Through the points of one of two coplanar similar ranges lines are drawn parallel to a given direction in the plane and through the corresponding points of the other range lines are drawn parallel to another given direction in the plane. Show that the intersections of corresponding lines lie on a fixed straight line.

[The points at infinity correspond. Take vertices X, Y of Art. 26 on line at infinity and result follows.]

4. All the vertices but one of a polygon lie on fixed lines, while its sides are parallel to fixed directions. Show that the locus of the last vertex is a straight line.

5. If the vertices of a polygon lie on fixed concurrent lines, while all the sides but one pass through fixed points, the last side also passes through a fixed point.

6. If the sides of a polygon pass through fixed collinear points, while all the vertices but one move on fixed straight lines, the locus of the last remaining vertex is a straight line.

7. Two concentric pencils are *oppositely* equal. Show that the two bisectors of the angles between any two corresponding rays are self-corresponding.

8. Three lines a, b, c meet at a point O, and D, E are fixed points not on any of these lines, nor in line with O. If points X, Y are taken on b, c respectively, so that DX, EY meet on a, show that the line XY passes through a certain fixed point of the line DE.

9. Two collinear projective ranges have a self-corresponding point A given and two pairs of corresponding points P_1, P_2; Q_1, Q_2. Show how to construct the second self-corresponding point, and prove that there cannot be more than one self-corresponding point, other than A.

10. Two concentric projective pencils in a plane have a self-corresponding ray a given and two pairs of corresponding rays p_1, p_2; q_1, q_2. Show how to construct the second self-corresponding ray.

11. A, B are two fixed points: P_1, P_2 are harmonically conjugate with regard to A, B. Show that the ranges $[P_1]$, $[P_2]$ are projective and find a geometrical construction by projections to pass from one to the other. What are the correspondents of the points A, B ?

12. Show that if $\{APBQ\}=\{AP'BQ'\}$, then $\{APBP'\}=\{AQBQ'\}$. Deduce that if A, B are self-corresponding elements of two collinear projective ranges, any two corresponding points determine with A, B a constant cross-ratio.

13. Prove that if

$$\{A_1B_1C_1P_1\}=\{A_2B_2C_2P_2\}$$
$$\{A_1B_1C_1Q_1\}=\{A_2B_2C_2Q_2\}$$
$$\{A_1B_1C_1R_1\}=\{A_2B_2C_2R_2\}$$
$$\{A_1B_1C_1S_1\}=\{A_2B_2C_2S_2\}.$$

then $$\{P_1Q_1R_1S_1\}=\{P_2Q_2R_2S_2\}.$$

14. Prove that if two corresponding ranges be such that any four elements of one have the same cross-ratio as the corresponding four elements of the other they are projective.

15. Given the cross-axis of two projective ranges and a pair of corresponding points, show how to construct the point of one range corresponding to a given point of the other. In particular construct the vanishing points.

16. Given the cross-centre of two projective pencils and a pair of corresponding rays, find a construction for the ray of one pencil corresponding to a given ray of the other.

17. A ray through a fixed point O cuts a line u at P_1 and the line at infinity at P_2^∞. P_1, P_2^∞ then describe projective ranges on u and on the line at infinity respectively. Show that the cross-axis of these two ranges is a parallel to u at a distance from u equal to the distance of O from u.

18. The arms OP, OQ of an angle of fixed magnitude which moves in one plane about its fixed vertex O intersect two given straight lines at P and Q respectively. Show that the ranges $[P]$, $[Q]$ are projective.

19. If in Ex. 18 one of the given straight lines is the line at infinity, construct the cross-axis of the ranges $[P]$, $[Q^\infty]$.

20. Through a point O a ray OPQ is drawn meeting two fixed lines at P, Q. If R be harmonically conjugate to O with regard to P, Q prove that the locus of R is a straight line.

21. A, B are two fixed points, u a fixed line. If P be any point of u and p be harmonically conjugate to u with regard to PA, PB, show that p passes through a fixed point.

22. Apply Menelaus' Theorem to prove Desargues' Theorem that if ABC, $A'B'C'$ be two coplanar triangles such that AA', BB', CC' are concurrent, then aa', bb', cc' are collinear and conversely.

23. ABC is a triangle, O any point in its plane. If OA meet BC at P, OB meet CA at Q, OC meet AB at R, and if P' be the harmonic conjugate of P with regard to BC, Q' the harmonic conjugate of Q with regard to CA, R' the harmonic conjugate of R with regard to AB, show that P', Q', R' are collinear. [$P'Q'R'$ is termed the **harmonic polar** of O with respect to the triangle.]

24. In Ex. 23 prove that the middle points of PP', QQ', RR' are collinear. Hence prove that the middle points of the diagonals of a quadrilateral are collinear.

25. ABC is a triangle, $A_1B_1C_1$ a transversal cutting BC, CA, AB at A_1, B_1, C_1. A_2 is the harmonic conjugate of A_1 with respect to B and C, B_2 is the harmonic conjugate of B_1 with respect to C and A. If AA_2, BB_2 meet at O, and CO meets AB at C_2, prove that C_2 is the harmonic conjugate of C_1 with respect to A and B.

26. Straight lines $AE'D'D$, $BF'E'E$ and $CD'F'F$ are drawn from the vertices A_1B_1C of a triangle meeting the opposite sides at D, E, F; and $BD:DC = CE:EA = AF:FB = 2:1$. Lines $AF'L$, $BD'M$ and $CE'N$ are drawn meeting the sides BC, CA, AB at L, M, N respectively. Prove that

$$DC:5DL = MC:3.ME.$$

27. The coplanar triangles ABC, PQR are in perspective, and D, E, F, G, H, I are the intersections (BC, PQ) (BC, PR) (CA, QR) (CA, QP) (AB, RP) (AB, RQ) respectively. Prove that

$$AF.AG.BH.BI.CD.CE = AH.AI.BD.BE.CF.CG.$$

EXAMPLES IIB

[The axes of co-ordinates are rectangular, except where otherwise stated.]

1. Draw two straight lines $OABCD$, $OA'C'B'$ making the angle $AOA' = 30°$; $OA = AB = BC = CD = 4$ cm., $OA' = 6$ cm., $A'C' = C'B' = 2$ cm. The points A, B, C correspond respectively to A', B', C' in two projective ranges.

Construct geometrically (i) the point corresponding to D, (ii) the vanishing point of the range on OA.

2. $O=(1, 0)$; $O'=(-1, 0)$; $A=(2, 3)$; $B=(\cdot 5, 2\cdot 5)$; $C=(-\cdot 5, 1)$; $D=(0, 4)$. Construct a ray $O'D'$ such that $O\{ABCD\}=O'\{ABCD'\}$.

3. $A_1B_1C_1D_1$, $A_2B_2C_2$ are given by distances from a fixed origin O equal to 2, 1, -3, 4; -1, 5, 2 respectively.
Construct geometrically a point D_2 such that

$$\{A_2B_2C_2D_2\}=\{A_1B_1C_1D_1\}$$

and verify your result by calculation.

4. A, B, C, D, E are five points in order on a straight line such that $AB=BC=CD=DE=\frac{1}{2}$ inch; and A, C, E correspond respectively to D, C, A in two projective ranges. Construct (i) the point of each range corresponding to B in the other, (ii) the vanishing points of the two ranges, and (iii) the second self-corresponding point.

5. Mark seven points A, A', B', B, C, C', D in order on a straight line so that $AA'=A'B'=B'B=BC=CC'=C'D=2$ cm. The points A, B, C correspond respectively to A', B', C' in two projective ranges; find by geometrical construction (i) the point D' of the second range to which D corresponds in the first, (ii) the vanishing point of the second range.

6. Construct the cross-axis of the ranges defined by the corresponding triads (0, 0), (0, 2), (0, 1); (1, 0), (0, 0), (3, 0) respectively, the axes of coordinates being inclined at 75°. Hence construct any pair of corresponding points of the ranges and the envelope of the joins of such points.

7. A, B, C are three points of a straight line, $AB=2$, $BC=1$. Construct points P, Q, R which shall be harmonically conjugate to A with respect to BC, B with respect to CA, C with respect to AB.

8. Construct a ray OD harmonically conjugate to OB with regard to OA, OC where the angles AOB, BOC are 30° and 15° respectively.

9. Using the ruler only, draw a line through a given point P and the inaccessible meet Q (not necessarily at infinity) of two straight lines a, b.

10. Given four rays through a point O, construct geometrically a segment of a straight line which shall measure the cross-ratio of the four rays in a given order. Hence show how to construct a segment which shall measure the cross-ratio of four points on a line.

11. Two projective ranges on the lines $x=0$, $x=2$ respectively, have as corresponding pairs of points (0, 0) and (2, 2), (0, 1) and (2, 2·5), (0, -1) and (2, ∞). Construct the envelope of the lines joining pairs of corresponding points on the two ranges.

CHAPTER III

THE CONIC

33. Definition of the conic. A **conic section** or **conic** is the projection of a circle, or the plane section of a cone (right or oblique) on a circular base.

Since in general a straight line meets a circle in two points, the same is true of a conic, because properties of incidence are unaltered by projection : and since from any point two tangents can in general be drawn to a circle, the same holds for the conic since properties of tangency are unaltered by projection.

It follows from the definition that any property of the circle which is projective, *i.e.* unaltered by projection, can be transferred at once to the conic.

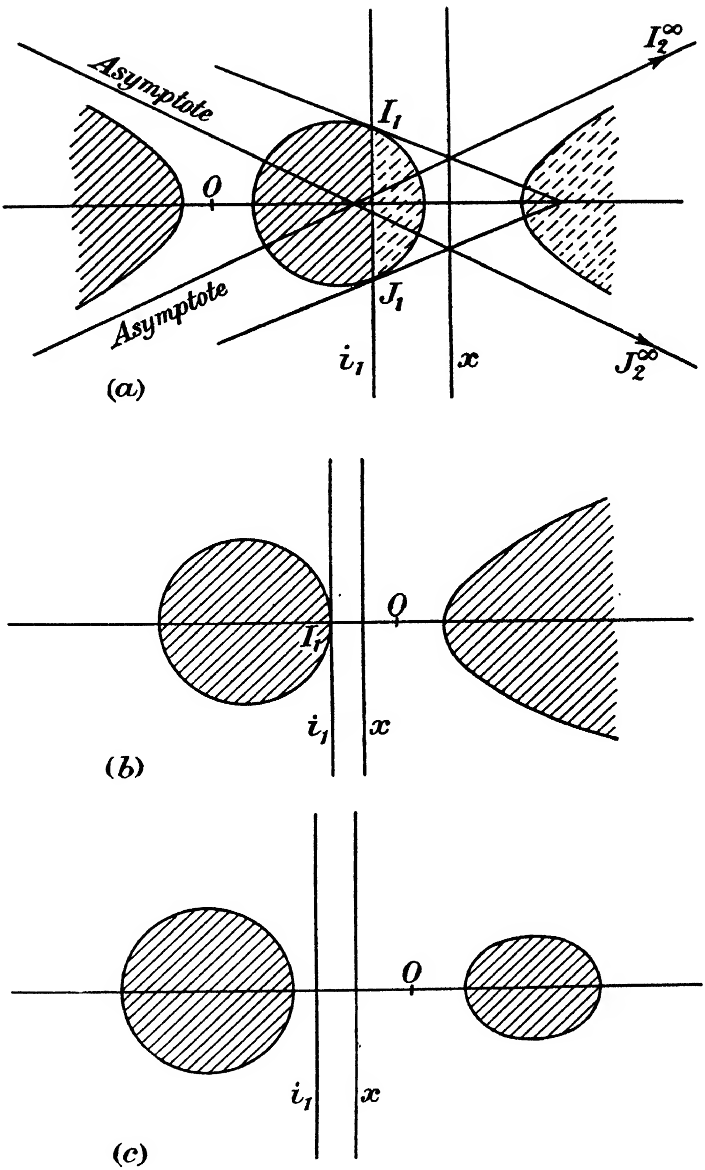
34. Types of conic. There are three types of conic, according as in the original figure the vanishing line cuts the circle in two real distinct points, or in two real coincident points (*i.e.* touches it) or does not cut it in real points. The three cases are shown in Fig. 16 (*a*), (*b*), (*c*).

In the first case (Fig. 16 (*a*)) there are two distinct points at infinity on the conic, namely the points I_2^∞, J_2^∞ corresponding to the intersections I_1, J_1 of the circle with the vanishing line. Such a conic is called a **hyperbola,** being shown as the rabatted projection of the circle (and therefore in plane perspective with it) in Fig. 16 (*a*).

The tangents to the circle at I_1, J_1 project into the tangents at I_2^∞, J_2^∞ to the conic. These two tangents are called the **asymptotes** of the conic. The curve has two branches, corresponding to the two parts into which the vanishing line divides the circle (Fig. 16 (*a*)).

If the vanishing line touch the circle (Fig. 16 (*b*)) I_1, J_1 coincide. The conic has two coincident points at infinity, *i.e.* it has the line at infinity for one of its tangents. Such a conic is called a **parabola.** It consists of one branch extending to infinity.

If the vanishing line do not cut the circle in real points (Fig. 16 (*c*)) there are no real points at infinity on the conic. The conic consists of an oval lying entirely at a finite distance and is called an **ellipse.**



I_2^∞
Asymptote
I_1
O
Asymptote
J_1
i_1
x
J_2^∞
(a)
O
I_1
i_1
x
(b)
O
i_1
x
(c)

FIG. 16.

The points which correspond in projection to the inside and outside of the circle are said to be inside and outside the conic respectively.

In the case of the ellipse and parabola, where the inside of the projected circle lies entirely on one side of the vanishing line, it projects into a single region. But, in the case of the hyperbola, the vanishing line divides the area interior to the circle into two segments, which are separated in projection. Each of these segments is bounded by a different branch of the curve. Corresponding regions inside the conic and circle are shown by similar shading in Fig. 16.

Since in cylindrical projection the vanishing line is at infinity the ellipse is the only one of the conics which can be obtained from the circle by cylindrical projection.

There are also two other types of conic, viz. the **line-pair** and **point-pair.** These are to some extent anomalous as, although they can be derived as limiting cases of projection of a circle, the reverse process is indeterminate. They will be discussed in Art. 44.

35. Curve as envelope and locus. The terms **locus** and **envelope** will frequently occur in what follows. A curve may be generated in two ways: (*a*) by a moving point P; we then speak of the curve as the locus of P and we construct it graphically from a large number of positions of P, forming a closely inscribed polygon; (*b*) by a moving tangent p; we then speak of the curve as the **envelope** of p and we construct it graphically from a large number of positions of p, forming a closely circumscribed polygon.

36. Chasles' Theorem. If P be a variable point on a circle, p the tangent at P, O any fixed point on the circle, t any fixed tangent to the circle, then the pencil $O[P]$ is **equi-anharmonic** with the range $t[p]$, that is, if P, Q, R, S be any four positions of P, p, q, r, s the corresponding tangents, then

$$O\{PQRS\} = t\{pqrs\}.$$

Let C (Fig. 17) be the centre of the circle, T the point of contact of t, $P' = pt$. Then $P'P$, $P'T$ being tangents to a circle, the angle $P'CT = \frac{1}{2}PCT =$ angle at the circumference POT.

Therefore by placing O on C and OT on CT the pencils $O[P]$, $C[P']$ are superposable. Hence they are directly equal. Hence by Art. 24

$$O[P] \barwedge C[P'],$$

$$\therefore\ O\{PQRS\} = C\{P'Q'R'S'\} = \{P'Q'R'S'\} = t\{pqrs\}.$$

37. Pencils obtained by joining a variable point of a circle to two fixed points. If O' be any other fixed point on the circle, we have by Chasles' Theorem

$$O'[P] \barwedge C[P'] \barwedge O[P].$$

Hence the joins of a variable point P on a circle to two fixed points O, O' on the circle sweep out projective pencils.

In the case of the circle these pencils are clearly *directly equal*, for the angle $TOP =$ angle $TO'P$ (Fig. 17), by the well-known property of angles in the same segment.

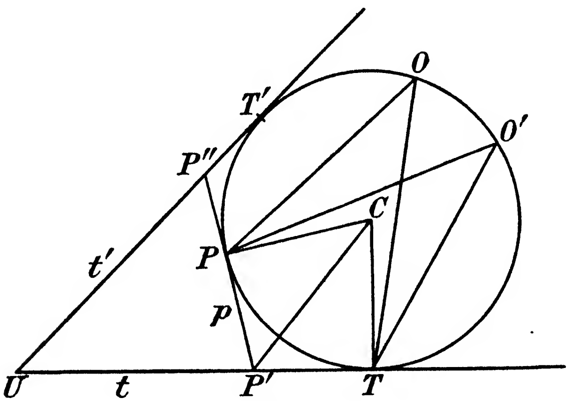


FIG. 17.

38. Ranges obtained by intersections of a variable tangent to a circle with two fixed tangents. Let t' (Fig. 17) be any other fixed tangent. Let $pt' = P''$. Then by Chasles' Theorem

$$C[P''] \barwedge O[P] \barwedge C[P'].$$

Cutting the projective pencils $C[P']$, $C[P'']$ by t, t'

$$[P'] \barwedge [P''],$$

or the intersections of a variable tangent p to a circle with two fixed tangents t, t' describe two projective ranges.

39. Corresponding properties for the conic. Since cross-ratios are not altered by projection and projective ranges and pencils project into projective ranges and pencils respectively, the properties stated in Arts. 36—38 hold for the conic, except that now the pencils will no longer be *equal*, for equal angles are not, in general, projected into equal angles. But the properties

$$O\{PQRS\} = t\{pqrs\} \qquad \text{(1)}$$

$$O[P] \barwedge O'[P] \qquad \text{(2)}$$

$$t[p] \barwedge t'[p] \qquad \text{(3)}$$

hold equally if for the word "circle" in the last three articles we read "conic."

From the property (2) it follows that every conic may be obtained as the locus of meets of corresponding rays of two projective pencils. From the property (3) it follows that every conic can be obtained as the envelope of joins of corresponding points of two projective ranges.

If in Fig. 17 P approaches O', OP approaches OO', $O'P$ approaches the tangent at O'. Hence to OO' considered as a ray of the pencil $O[P]$ corresponds the tangent at O'. Similarly to $O'O$ considered as a ray of the pencil $O'[P]$ corresponds the tangent at O. The cross-centre (Art. 30) of the two pencils through O, O' is therefore the point of intersection of the tangents at O, O'.

Again, if p approaches t, P' approaches T and P'' approaches the intersection U of the two tangents t, t'. Hence to tt' considered as a point of range $t'[p]$ corresponds the point of contact T of t. Similarly to tt' considered as a point of range $t[p]$ corresponds the point of contact T' of t'. The cross-axis of the two ranges is therefore the chord of contact TT' (Art. 30).

In the above reasoning it is immaterial whether the curve of Fig. 17 be a circle or a conic.

Examples

1. Show that with four points A, B, C, D on a conic may be associated a definite cross-ratio: and also that with four tangents a, b, c, d may be associated a definite cross-ratio; and show that the cross-ratio of four such tangents is the cross-ratio of their four points of contact.

2. A variable tangent LL' meets two fixed parallel tangents to a conic at L, L'; if A, B are the points of contact of the fixed tangents, prove that $AL.BL'$ is constant.

3. A variable tangent meets the asymptotes of a hyperbola at P, P'. If C be the intersection of the asymptotes, prove that $CP.CP' =$ constant.

40. Property of tangents to a parabola. The property (3) of the last article takes a particularly simple form when the conic is a parabola. For then the line at infinity is a tangent to the curve by Art. 34. Hence the points at infinity of the ranges t, t' correspond, and by Art. 24 the ranges are similar. But T, P', U correspond to U, P'', T' : hence

$$TP' : P'U = UP'' : P''T',$$

or the intercepts made by a variable tangent to a parabola on two fixed tangents are inversely proportional. This furnishes an easy graphical method of drawing a parabola as an envelope, two tangents UT, UT' and their points of contact T, T' being given. n being a large integer, take lengths $\frac{1}{n}UT'$ and $\frac{1}{n}UT$ and lay them off in succession any number of times upon UT', TU respectively, starting from U along UT' and from T along TU. Join corresponding points of division; each of these is a tangent to the parabola.

Example

OT, OV are tangents to a parabola whose points of contact are T and V. Show that the tangent to the parabola parallel to TV bisects OT, OV.

41. The product of any two projective pencils is a conic. We shall call the locus of meets of corresponding rays of two pencils the **product** of the pencils and the envelope of joins of corresponding points of two ranges the **product** of the ranges.

We have seen that every conic can be obtained as the product of two projective pencils. But these pencils might be projective pencils of a *special* type (as in the case of the circle, where they are *equal*). We will now show that any two projective pencils whatever lead to a conic locus.

Let $S[P]$, $O[P]$ (Fig. 18) be the two pencils and let OT be the ray of the pencil O corresponding to SO of the pencil S. Draw any circle touching OT at O. Let this circle meet OP at P', OS at S'. By Art. 37

$$O[P'] \barwedge S'[P'].$$

But $$O[P'] \equiv O[P] \barwedge S[P],$$

$$\therefore\ S[P] \barwedge S'[P'].$$

Also $S'O$ of pencil $S'[P']$ corresponds in the pencil $O[P']$ to the tangent OT at O, and this in turn corresponds to SO. Hence in pencils $S[P]$, $S'[P']$ the ray SS' is self-corresponding. Therefore by Art. 26 $S[P]$, $S'[P']$ are perspective, therefore corresponding rays SP $S'P'$ meet at X on a fixed line x. P is therefore constructed from P by the construction for two figures in plane perspective, O being the pole and x the axis of perspective, S and S' a given pair of corresponding points. For PP' passes through O and SP, $S'P'$ meet on x.

The locus of P is thus in plane perspective with the circle which is the locus of P', that is, it may be looked upon as the rabatted projection of this circle upon another plane. It is therefore a conic by definition. Note that the conic and circle touch at O; if they intersect again at Y, Z, then Y, Z must be self-corresponding points and the axis of perspective x passes through them.

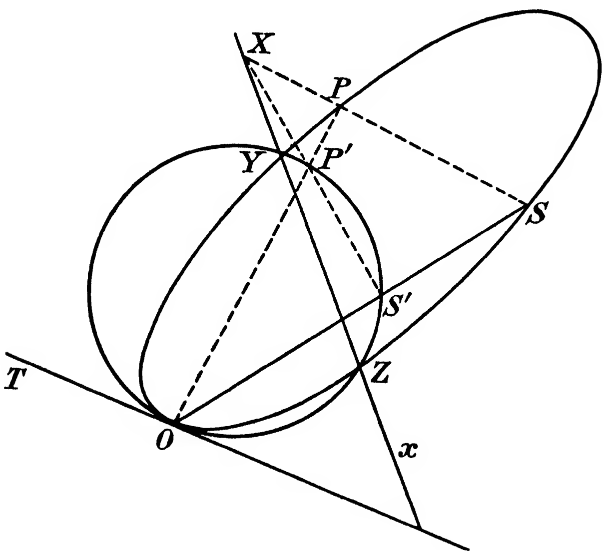


FIG. 18.

42. The product of any two projective ranges is a conic. Let $[P]$, $[P_0]$ (Fig. 19) be the ranges, t, x their bases, $p = PP_0$. Let T be the point of range x corresponding to the intersection U of t, x. Draw any circle touching x at T and from U, P_0 draw tangents t', p' to this circle. Let $p't' = P'$. Then $[P'] \barwedge [P_0] \barwedge [P]$. Also U of range $[P']$ corresponds to T of range $[P_0]$ and T of range $[P_0]$ corresponds to U of range $[P]$. Hence the ranges $[P]$, $[P']$ have a self-corresponding point U. They are therefore perspective ranges by Art. 26. Hence PP' passes through a fixed point O. The lines p', p are obtained from one another by the construction for figures in plane perspective, O being the pole, x the axis, t, t' a pair of given corresponding lines. For p, p' meet on x and $(pt, p't')$ passes through

O. Hence the envelope of p is in plane perspective with the envelope of p' and, as in the last article, must be a conic.

The conic and circle both touch x at T. If they have other real common tangents y, z, these must be self-corresponding lines and so pass through O.

43. Deductions from the above. In the proofs of Arts. 41, 42 the circle may clearly be replaced by any conic. For the only properties of the circle made use of in the proofs are also, by Art. 39, properties of the conic.

It follows that two conics in contact can be brought into plane

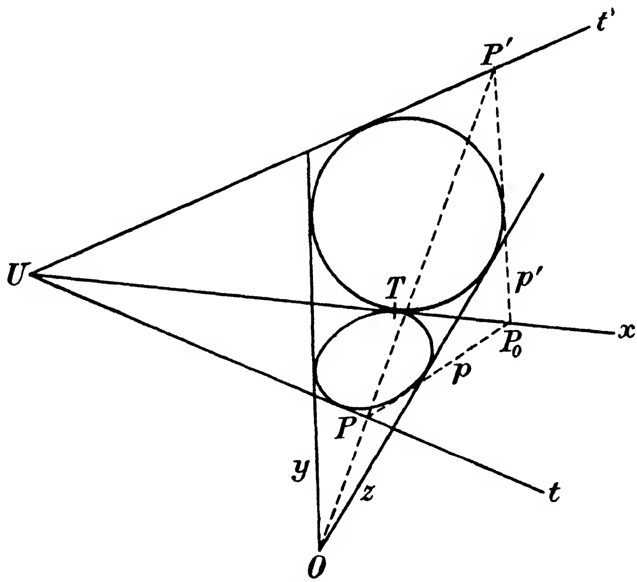


Fig. 19.

perspective in two ways, viz. (1) by taking the point of contact to be the pole of perspective: the axis of perspective is then a line passing through the remaining two intersections of the two conics; (2) by taking the common tangent at the point of contact to be the axis of perspective: the pole of perspective is then a point through which pass the remaining common tangents of the two conics.

These, however, are not the only ways in which such conics can be brought into plane perspective (*cf.* Exs. IIIA. 11, 13).

Note also that the product of two projective pencils passes through the vertices of the pencils.

Thus the product of two projective pencils of parallel rays is a hyperbola whose points at infinity and therefore the directions of whose asymptotes are given by the directions of the two pencils.

Similarly the product of two projective ranges touches the bases of the ranges. Thus the product of a range on an accessible base and a projective range on the line at infinity is a parabola.

If therefore through a fixed point O a ray OP be drawn meeting a fixed line u at P, and PQ^∞ be drawn through P making a fixed angle α with OP, PQ^∞ touches a fixed parabola. For draw OQ^∞ parallel to PQ^∞, the pencils $O[P]$, $O[Q^\infty]$ are superposable by means of a rotation α about O. They are therefore equal and projective. Hence the ranges $[P]$, $[Q^\infty]$ are projective. The latter being on the line at infinity, the result stated follows.

It will also follow from Art. 41, since projective pencils project into projective pencils, that the projection of a conic is a conic.

Examples

1. On the tangent at O to a conic any point P is taken and PT is drawn to touch the conic at T. If S be any other fixed point on the conic, show that the locus of the intersection of OT, SP is another conic, which touches the original conic at O and S.

2. O, O' are two fixed points on a conic s; l is a fixed straight line. P is any point on s; OP, $O'P$ are joined, meeting l at R, R' respectively; OR', $O'R$ meet at Q. Show that the locus of Q is another conic.

3. O, S are fixed points, a, b fixed straight lines. A line through O meets a at P, b at Q. Through Q is drawn a parallel to SP. Show that this parallel touches a fixed parabola.

4. Two conics touch one another at O. From a point on the common tangent at O lines are drawn to touch the conics at P, P'. Show that PP' passes through a fixed point, through which passes any other common tangent to the two conics.

5. Two conics touch one another at O. A line through O meets the conics at P, P'. Show that the tangents at P, P' meet on a fixed line, which is that common chord of the conics which does not pass through O, if such a common chord exists.

6. If, in Ex. 5 the conics are circles, prove that the tangents at P, P' are parallel.

7. ORP, OSQ and PQ are fixed straight lines, and A, B are two fixed points. Straight lines RS are drawn parallel to PQ. Prove that AR and BS meet on a conic which passes through A, B, O.

Show how to determine the tangents at A and B to this conic.

44. Line-pair and point-pair. If the two projective pencils of Art. 41 are *perspective* their product breaks up into two straight lines, namely the axis of perspective and the self-corresponding ray, since the latter may be regarded as intersecting itself at any one of its points, and therefore must figure in the locus of intersection of corresponding rays.

A **line-pair** is therefore a special case of a conic locus.

If the two projective ranges of Art. 42 are perspective, their product breaks up into two sets of lines, one passing through the vertex of perspective, the other passing through the self-corresponding point, since any ray through the latter may be looked upon as the join of the point to itself. The envelope then reduces to these two points, so that:

A **point-pair** is a special case of a conic envelope.

The line-pair and point-pair present certain anomalies which should be noticed.

Let the components of a line-pair be a, b and their meet C. Then a line through a point P of the plane meets the line-pair in two distinct points, unless the line passes through C when the intersections coincide. Thus from any point one tangent, and one only, can be drawn to a line-pair. We may, to keep the properties of conics perfectly general, look upon this as two coincident tangents, but it is then no longer true that a point, the two tangents from which to a conic are coincident, is itself on the conic.

On the other hand, let the components of a point-pair be A, B and their join c. From a point P two distinct tangents PA, PB can be drawn to the point-pair, except if P be on c when they coincide. The points of c may therefore be looked upon as belonging to the point-pair conic. Any straight line then meets the point-pair in one point, and one only. If we look upon this point as two coincident points, to preserve the property that a straight line meets a conic in two points, it will appear that a straight line can meet a conic in two coincident points, without being a tangent to it, for lines not through A and B are not tangents to the point-pair.

The true significance of the line-pair and point-pair will be more apparent later on when we come to study the central and focal properties of the conics.

It is interesting to note the manner in which the line-pair and point-pair appear as projections of a circle. To obtain the line-pair, project the circle from a vertex V outside its plane and cut the cone so formed by a plane through V, *i.e.* take the vertex *in* the plane of projection. Thus all the points of the circle project into V (which is then the intersection of the lines of the pair), except the two points where the circle meets the plane of projection, which points project into any point of the corresponding line of the pair.

To obtain the point-pair take V in the plane of the circle, but outside the circle, and project on to any other plane. In this case, all the tangents to the circle project into the line joining the points of the pair, with the exception of the two tangents through V, which project into the points of the pair.

It should be noticed that, if we try to reverse these projections, the process is indeterminate. Thus the points of a line-pair, other than the double point V, project from V on to any plane into two definite points, but V itself may project into any point of the plane. Similarly the tangents to a point-pair A, B, other than those passing through V, project upon the plane VAB into any line of that plane, but those through V project into two definite straight lines VA, VB. We cannot therefore obtain the circle *from* the line-pair or point-pair by projection.

Note also that, in these cases, the line-pair, or point-pair, cannot be brought into plane perspective with a circle.

45. A conic is determined by five points or by five tangents. For let O, O', A, B, C be five points on a conic, P any sixth point, then the pencils

$$O(ABCP),\ O'(ABCP)$$

are projective. But O, O', A, B, C determine completely the corresponding triads $O(ABC)$, $O'(ABC)$, and these in turn determine completely the relation between the pencils. Hence OP being given, $O'P$ is known, that is, P is determined. Every point on the conic is therefore fixed when five points are fixed. It follows that two distinct conics cannot have more than four points of intersection.

The points A, O (and also the points B, O') may coincide without making the constructions indeterminate, provided we interpret OA, $O'B$ as the tangents at O, O'. Accordingly being given a point on the conic and the tangent at this point is equivalent to being given two points.

Similarly if t, t', a, b, c be five tangents to a conic, p any sixth tangent, pt, pt' are corresponding points of the projective ranges defined by the triads $t\,(abc)$, $t'\,(abc)$. Therefore when pt is known, pt' is known, and p is determined. Thus every tangent is determinate when five are given. It follows that two distinct conics cannot have more than four common tangents.

As before a, t (and also b, t') may coincide without making the constructions indeterminate, provided we interpret at, bt' as the points of contact of t, t'. Thus being given a tangent and its point of contact is equivalent to being given two tangents.

From the above we obtain a construction for the conic passing through five given points. For take two of the given points for vertices of two projective pencils and obtain corresponding triads by joining to the three remaining points. Construct pairs of corresponding rays by the method of Art. 26, or any other. The intersections of corresponding rays are points on the conic.

A precisely similar method can be applied to obtain a conic as an envelope when five tangents are given.

EXAMPLE

Given three points on a hyperbola and the directions of its asymptotes, show how to construct the pencils of parallel rays which generate the hyperbola and deduce a construction for the asymptotes.

46. Conics having three-point or four-point contact. The constructions of Art. 43 lead to important particular cases.

In the first place it is clear that, if we take a point O of a conic s_1 as pole of perspective, and any line x, meeting s_1 at X, Y, as axis of perspective, we obtain as the curve corresponding to s_1 a conic s_2 passing through O, X, Y. Since lines through O are self-corresponding, s_1, s_2 have a common tangent at O, so that they can have no other intersections.

If we now make X approach O, x ultimately passes through O, and s_1, s_2 have *three* coincident intersections at O; they are then said to have **three-point contact** at O. Thus two conics having three-point contact at O can be brought into plane perspective by taking O and the common chord as pole and axis of perspective.

Since five points determine a conic, it is in general possible to construct a conic s_2 having three-point contact with s_1 at O, and passing through two given points P_2, Q_2 of the plane. To do this, join OP_2, OQ_2; these must meet s_1 at the corresponding points P_1, Q_1. The chords P_1Q_1, P_2Q_2 meet at X on the axis of perspective. Since this must pass through O, it is OX, and is determined. s_2 can then be constructed by the methods of Art. 16.

If, however, s_2 is a circle having three-point contact with s_1 at O, only one such circle exists, since a circle is entirely determined by three points. s_2 is then said to be the **circle of curvature** at O. Its centre and radius are termed the **centre** and **radius of curvature** at O.

Returning to the more general case of two conics, let now Y also approach O, then x approaches the tangent at O, and the four intersections of the two conics coincide at O. If, therefore, we transform a conic s_1 by a plane perspective, using a point O of s_1 as pole and the tangent x at O to s_1 as axis of perspective, we obtain a conic s_2 which has **four-point contact** with s_1 at O.

We can always construct one conic s_2 having four-point contact with a given conic s_1 at O and passing through a given point P_2 of the plane. For if OP_2 meets s_1 at P_1, we have the pole O of perspective, the axis of perspective, namely the tangent at O, and a pair of corresponding points P_1, P_2. Unless O is a special point on the conic, it is in general impossible to select P_2 so that s_2 is a circle.

There exists one parabola s_2, however, which has four-point contact with s_1 at a given point O. For clearly, in the perspective transformation from i_1 to s_2, since the line at infinity touches s_2, the vanishing line s_1 must touch s_1. Because the vanishing line is parallel to the axis of perspective, it must in this case be the tangent to s_1 parallel to the tangent at O. This defines completely the perspective relation, since we now have one vanishing line together with the pole and axis of perspective. We can therefore construct s_2. In particular the point at infinity on the parabola is on the line joining O to the point of contact of the tangent to s_1 parallel to the tangent at O.

Since a conic is determined by five points, it is impossible to have higher than four-point contact between two distinct conics.

A curve which has with another contact of the highest possible order is said to **osculate** it. Thus the circle of curvature is sometimes spoken of as the osculating circle ; the parabola having four-point contact with a conic as the osculating parabola, and so on.

47. Conics having three-line or four-line contact. If, in a similar manner, we take as axis of perspective the tangent c at C to a conic s_1, and as pole of perspective a point O outside s_1, the tangents from O to s_1 being x, y, we obtain, since x, y, c, C are self-corresponding, a conic s_2 touching c at C and touching x, y. The conics s_1, s_2 have thus two common tangents coincident with c, and two other common tangents x, y, so that they can have no other common tangents.

If now x coincides with c, we have *three* common tangents coincident with c, and the conics are said to have **three-line contact** at C. The pole O of perspective is then the intersection of c with the single remaining common tangent y.

If, further, y now approaches c, O approaches C, and we have in the limit two conics having **four-line contact** at C, and these are in plane perspective, the common tangent c at C being the axis of perspective, and C itself the pole of perspective.

We will now show that, **if two conics s_1, s_2 have three-line contact at C, they also have three-point contact at C.**

As in Art. 41, draw through C any fixed line CA_1A_2 and a variable line CP_1P_2, meeting s_1, s_2 at A_1, A_2 and P_1, P_2 respectively. Then $A_1[P_1]\barwedge C[P_1]\equiv C[P_2]\barwedge A_2[P_2]$ and if CP_1P_2 is drawn along the tangent at C, P_1 and P_2 coincide at C, and in the projective pencils $A_1[P_1]$, $A_2[P_2]$, A_1C corresponds to A_2C so that CA_1A_2 is a self-corresponding ray. The pencils are accordingly perspective, and corresponding rays meet on a line u. If now u, C are taken as axis and pole of perspective, A_1 and A_2 as a pair of corresponding points, s_1 will transform into s_2 by the usual construction (Art. 16).

But we have seen that s_1 and s_2 also correspond in a second perspective, in which c is the axis and O is the pole. Let the axes u, c of the first and second perspective meet at X.

Now if from any point on an axis of perspective tangents are drawn to two corresponding conics, these tangents must necessarily correspond in pairs, since corresponding tangents meet on the axis.

In the given case, if t_1, t_2 are the tangents from X to s_1 and s_2 (other than the common tangent c), c is self-corresponding in both perspectives (since it passes through both poles) and therefore t_1, t_2 are corresponding lines in both perspectives. Hence their points of contact T_1, T_2 are corresponding points in both perspectives, so that T_1T_2 passes through both O and C. Thus T_1, T_2 lie on c, and therefore must coincide with C; and X must then also coincide with C.

Hence the axis u of the first perspective passes through C. But, by Art. 46, this is the condition that the conics s_1, s_2 shall have three-point contact. This proves the theorem. Thus no distinction need be made between three-line and three-point contact.

In the case of conics having four-line contact, it will be noticed that the plane perspective relation connecting them is of the same form as that connecting two conics having four-point contact. Here again, then, four-line contact implies four-point contact and conversely.

Examples

1. Prove that, if two conics have three-point contact at C, they also have three-line contact at C.

2. Prove that one conic can be constructed having three-point (or three-line) contact with a given conic at C, and touching two given straight lines in the plane.

3. Show that, in the plane perspective relation between a parabola and its circle of curvature at P, in which the common tangent at P is the axis of perspective, the tangent to the circle parallel to the tangent at P, treated as a line of the parabola figure, corresponds to the parallel diameter of the circle, in the circle figure.

4. The asymptotes of a hyperbola meet at C. If O is the pole of the plane perspective which transforms the hyperbola into its circle of curvature at P, the tangent at P being the axis of perspective, and the diameter of the circle through P meets the vanishing line of the circle at I, prove that OI is parallel to CP.

5. If, in Ex. 4, the tangent at P meets an asymptote at D, and CN is the perpendicular from C on the tangent at P, prove that

$$PI = 2PD^2.CN/(PD^2 + CN^2).$$

EXAMPLES IIIA

1. O, O', A, B, C, D are six points on a conic. If $(OA, O'B) = P$, $(OB, O'C) = Q$, $(OC, O'D) = R$, $(OD, O'A) = S$, prove that if P, Q, R, S, O, O' lie on a conic the rays OB, OD are harmonically conjugate with regard to OA, OC.

2. A, B, C, D, U, V are six points on a conic; prove that (UA, VC), (UB, VD), (UC, VA), (UD, VB) lie on a conic passing through U and V.

3. The tangents to a conic s at points A and B meet at T, and a variable tangent meets TA, TB at X, Y respectively. If the parallelogram $TXYZ$ is completed, show that the locus of Z is the hyperbola through A and B whose asymptotes are the tangents to s parallel to TA and TB.

4. The arms OA, OB of an angle of fixed magnitude α and of fixed vertex O, meet a fixed straight line at A and B, and through A and B respectively lines AP, BP are drawn parallel to fixed directions. Show that the locus of P is a hyperbola and find its asymptotes.

5. Two oppositely equal pencils have two different vertices. Show that their product is a rectangular hyperbola (*i.e.* one whose asymptotes are at right angles). [Find when corresponding rays of the two pencils are parallel.]

6. Through two fixed points A, B pairs of parallel lines AP, BQ are drawn to meet two fixed intersecting lines c, d at P, Q respectively. If cd is not in line with A and B, show that PQ touches a fixed conic to which c and d are tangents; and find the points of contact of c and d. How is this result modified if A, B and cd are collinear?

7. A tangent to a conic at P meets a fixed tangent at Q and QR is drawn through Q parallel to OP where O is a fixed point on the conic. Show that QR touches a parabola.

8. The sides of a polygon pass through fixed points and all the vertices but one lie on fixed lines. What is the locus of the last remaining vertex?

9. The vertices of a polygon lie on fixed lines and all the sides but one pass through fixed points. What is the envelope of the last remaining side?

10. AX, BY are drawn perpendicular to a given line AB. Two points P and Q trace out ranges in perspective on AX and BY respectively. Prove that the locus of intersections of AQ and BP is a hyperbola, and find the directions of its asymptotes.

11. Show that any two conics s_1, s_2 in a plane can be brought into a plane perspective relation by taking one of their common chords as axis of perspective. [Let XY be the common chord, A_1A_2 a common tangent touching s_1 at A_1, s_2 at A_2; Z a point on XY: let A_1Z meet s_1 at B_1, A_2Z meet s_2 at B_2. Take $O=(B_1B_2, A_1A_2)$; then the perspective relation defined by pole O, axis XY and pair of corresponding points A_1, A_2 transforms s_1 into a conic through the three points X, Y, B_2, and touching OA_2 at A_2, *i.e.* into s_2.]

12. From any point on one of their common chords tangents are drawn to two conics, touching the conics at P, Q. Show that PQ passes through one of two fixed points.

13. Show that any two conics s_1, s_2 in a plane can be brought into a plane perspective relation by taking the meet of two of their common tangents as pole of perspective.

14. Through the meet O of two common tangents to two conics a line is drawn meeting one conic at P and the other at Q. Show that the tangents at P, Q meet on one of two fixed lines.

15. A, B are two fixed points on any circle. Show that a point O and a straight line c exist in the plane of the circle, such that, if R be any point of the circle, and AR, BR meet c at P, Q, the angle POQ is equal to a given angle.

[O is the pole of perspective and c the vanishing line when the circle on AB, containing the given angle, is brought into plane perspective with the given circle, with AB as axis of perspective.]

16. Prove that the envelope of the harmonic polars (*see* Exs. IIA, 23), with regard to a triangle ABC, of the points of a line u, not passing through A, B, or C, is a conic which touches BC, CA, AB at points P, Q, R such that AP, BQ, CR concur at the point U whose harmonic polar with respect to the triangle is u.

17. Prove that if ABC, $A'B'C'$ be two triangles inscribed in a conic, their six sides touch another conic.

[For $B(ACA'C')\barwedge B'(ACA'C')$; cut these pencils by $A'C'$ in $(DEA'C')$ and by AC in $(ACFG)$ respectively; then $(DEA'C')\barwedge(ACFG)$ and by Art. 42, $AD=AB$, $CE=BC$, $A'F=A'B'$, $C'G=B'C'$ touch a conic which touches the bases AC, $A'C'$ of the ranges.]

18. Prove that if abc, $a'b'c'$ be two triangles circumscribed to a conic, their six vertices lie on a conic.

19. Deduce from Exs. 17, 18 Poncelet's Theorem that if there exist one triangle inscribed in one conic and circumscribed to another, there exist an infinity of such triangles.

EXAMPLES IIIB

[The axes of co-ordinates are rectangular, except where otherwise stated.]

1. There are two projective pencils of rays whose centres are at the points (1, 2) and (4, 6). The rays

$$y-2=0,\ 2x-y=0,\ x-y+1=0$$

of the first pencil correspond to the rays

$$x-y+2=0,\ 2x-3y+10=0,\ x-4=0$$

respectively of the second pencil.

Construct the ray of the second pencil corresponding to the ray

$4x-3y+2=0$ of the first pencil. Construct also the tangents at the points (1, 2) and (4, 6) to the conic which is the product of the pencils.

2. Two projective pencils of rays have their centres at (0, 0) and (1, 0) respectively. The rays $y-2x=0$, $y-x=0$, $3y-2x=0$ of the first pencil correspond to the rays $x-1=0$, $3y-4x+4=0$, $4y-3x+3=0$ respectively. Construct the ray of the second pencil corresponding to the ray $y+2x=0$ of the first pencil and sufficient points to exhibit the shape of the conic which is the product of the pencils.

3. AB, AC are two lines inclined at 60°. $AB=2''$; $AC=4''$. Construct by tangents the parabola which touches AB, AC at B and C.

4. Two projective pencils of parallel rays are given by the triads $x=0$, 1, 3, $y=0$, -2, 1, the axes being inclined at 60°. Draw the locus of intersections of corresponding rays of the pencils and construct its asymptotes.

5. Draw any two circles in contact. Verify that if tangents be drawn to the circles from any point on their common tangent, the join of the points of contact passes through a fixed point.

6. Given circles of radii 1 and 2 inches respectively, whose centres are 4 inches apart, construct a point O, a straight line x and a pair of points A, A' such that, in the plane perspective relation defined by O as pole of perspective, x as axis of collineation and A, A' as a pair of corresponding points, the two given circles are corresponding curves.

7. The axes of co-ordinates being inclined at 45°, two projective ranges on these axes have the points (5, 0) and (0, 4) for vanishing points and the points (3, 0), (0, 3) for a pair of corresponding points. Construct by tangents the product of these ranges.

8. Draw the conic which passes through the points

$$(0, 0), (4, 0), (2, 2), (3, 2), (0, -2).$$

9. A hyperbola has one asymptote parallel to the axis of x, touches the line $x+y=4$ at the point (2, 2), and passes through the points (2, 4) and (3·25, 1·5). Draw the curve.

10. Two parabolas have the circle $x^2+y^2=2$ for circle of curvature at the point (1, 1) and cut this circle again at the point $(1, -1)$. Construct their tangents at $(1, -1)$. Find also the directions of the points at infinity on the two parabolas and construct the tangents perpendicular to these directions and their points of contact. [The tangents to the circle parallel to the common chord of curvature give two possible vanishing lines.]

11. An equilateral triangle ABC is inscribed in a circle of radius 2 inches, and B is the middle point of AD. Find the points in which BC meets the conic through D which has four-point contact at A with the circle ABC.

CHAPTER IV

POLE AND POLAR

48. Polar of a point with regard to a conic. Let O be a fixed point in the plane of a given conic, and OAB, OPQ (Fig. 20) two chords meeting the conic at A, B and P, Q respectively.

If we consider the conic as the product of projective pencils through A and B, then AP, BP and AQ, BQ are corresponding pairs in the pencils.

If $(AQ, BP) = U$, and $(AP, BQ) = V$, then UV passes through the cross-centre K of the two pencils, which is also the intersection of the tangents at A and B (Art. 39).

But, by the harmonic property of the complete quadrangle $ABPQ$, VO, VU are harmonically conjugate with regard to VA, VB.

Therefore, if OAB, OPQ meet UV at C, R respectively, C is harmonically conjugate to O with respect to A, B; and R is harmonically conjugate to O with respect to P, Q.

Now let the chord OPQ turn round O, so that P, R, Q vary. Since O, A, B are fixed, K and C are fixed, hence KC (*i.e.* UV) is a fixed line l. Thus R describes a fixed straight line, which is termed the **polar** of O with respect to the conic.

Since there can be, on a given chord OPQ, only one point R harmonically conjugate to O with respect to P, Q, the polar of O is uniquely determined.

As OPQ is any chord through O, any such chord is harmonically divided by O and by its polar. If P and Q coincide, R coincides with them; hence the polar of O passes through the points of contact T, S of tangents from O to the conic, when real tangents can be drawn, that is, when O is outside the conic.

Note also that OAB is an arbitrary chord through O, and the tangents at A and B meet at K on the polar of O. By symmetry the tangents at P and Q also meet at N on the polar of O.

Further, since $\{OACB\} = -1$, $K\{OACB\} = -1$, or KC is harmonically conjugate to KO with regard to the two tangents KA, KB to the conic from K.

It will be noticed that the definition of the polar, as the locus of the point R, which is harmonically conjugate to O with regard to P, Q, ceases to be operative for the points of the polar on lines through O which do not meet the conic in real points. Such a line through O must lie entirely outside the conic, so that a point K, at which it meets the polar of O, lies outside the conic, and real

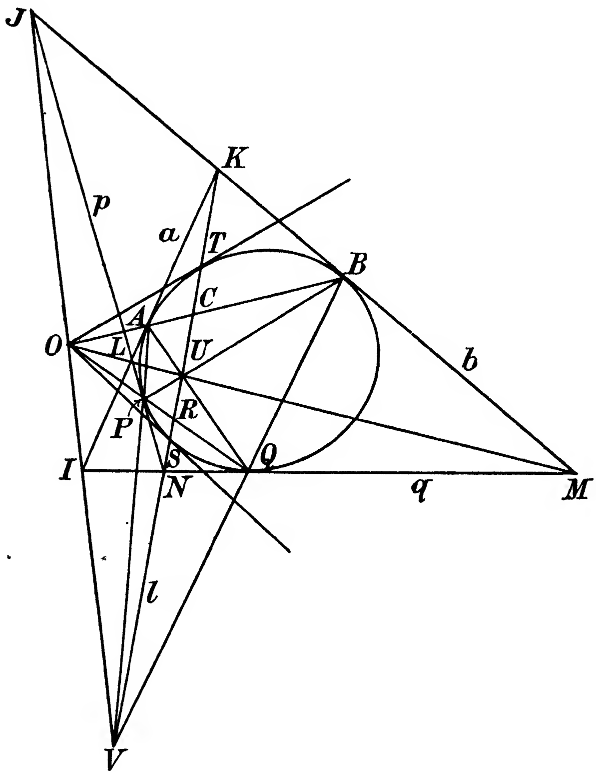


FIG. 20.

tangents KA, KB can be drawn from K to the conic. The points of the polar of type K can then be obtained as the intersections of tangents at the extremities of a chord OAB through O which does meet the conic in real points.

In Fig. 20, O has been taken outside the conic, but the argument remains the same if O be inside. In this case, however, every chord through O meets the conic and all the points on the polar are outside the conic. There is here no distinction between points of type R and points of type K.

Since the chord of contact of tangents to the conic from any point on l of type K passes through O, it is evident that O is uniquely determined by the intersection of such chords, when l is given. The point O is termed the **pole** of l with respect to the conic, and any line l which is not a tangent to the conic is the polar of a definite point not lying on l.

An important case occurs when O is on the conic. In this case, if OPQ be drawn to cut the conic, one of P, Q coincides with O, so that R coincides with O. But if OPQ be drawn to touch the conic, both P and Q coincide with O, and R may be any point on OPQ. Thus the locus of R is the tangent at O. This is also otherwise seen if we remember that the polar of O is the chord of contact TS of the tangents from O; if O approaches the conic, the tangents OT, OS and the chord TS ultimately coincide with the tangent at O.

Conversely, if l is a tangent to the conic, touching the curve at O, and K is any other point on l, the chord of contact of tangents from K passes through O, which is thus the pole of l.

Thus the polar of a point on the conic is the tangent at that point, and the pole of a tangent to the conic is its point of contact.

The relation of pole and polar is therefore a unique one; any point O has a unique polar l, and any line l has a unique pole O. If O lies on l, then O is on the conic and l is the tangent at O.

Examples

1. Show that the cross-centre of two projective pencils is the pole with regard to their product of the join of their vertices.

2. Show that the cross-axis of two projective ranges is the polar with regard to their product of the meet of their bases.

3. Given a point and its polar with regard to a conic and a point on the curve, find another point on the curve.

4. Given a point and its polar with regard to a conic and a tangent to the curve, find another tangent to the curve.

49. Conjugate points and lines. Let OC (Fig. 20) be a line through O meeting the conic in real points A, B, and the polar of O at C. Then O, C are harmonically conjugate with respect to A, B. But this also expresses the condition that O is a point on the polar of C. Hence if C is on the polar of O, O is on the polar of C.

Again let OK be a line through O which does not meet the conic, but meets the polar of O at K. Then Fig. 20 shows that, if KA, KB are the tangents from K to the conic, AB passes through O. But AB is the polar of K. Hence again, if K is on the polar of O, O lies on the polar of K.

Two points which have the property that either lies on the polar of the other are said to be **conjugate points** with respect to the conic.

It is obvious that two conjugate points are harmonically conjugate with regard to the two intersections of their join with the conic, when the join in question meets the conic.

Let f be the polar of a point F, g the polar of a point G. We have seen that if f passes through G, g passes through F. Two lines f, g, which are such that either passes through the pole of the other, are said to be **conjugate lines** with respect to the conic.

Obviously, in the above, F, G satisfy the condition for conjugate points. Hence the poles of conjugate lines are conjugate points and conversely, the polars of conjugate points are conjugate lines.

In Fig. 20, K being any point on the polar of O outside the conic, KO and the polar of O are conjugate lines, and we have seen that they are harmonically conjugate with regard to the tangents KA, KB from K.

Hence two conjugate lines are harmonically conjugate with regard to the two tangents from their meet to the conic, when the meet in question is outside the conic.

50. Inscribed quadrangle and circumscribed quadrilateral. Self-polar triangle. If we refer again to Fig. 20, it will be noticed that O, U, V are the three diagonal points of the complete quadrangle $ABPQ$ inscribed in the conic. We have proved that UV is the polar of O. But clearly, we might equally well have started by drawing chords AUQ, BUP through U, or chords VPA, VQB through V. We should then have proved that OV is the polar of U, or that OU is the polar of V.

The triangle OUV is therefore such that any two of its vertices, or any two of its sides, are conjugate with respect to the conic, and each side is the polar of the opposite vertex. Such a triangle is said to be **self-polar** with respect to the conic.

Consider now the tangents a, b, p, q at A, B, P, Q. These form a complete quadrilateral, of which the six vertices are I, J, K, L, M, N (Fig. 20). Clearly, by what has been proved in Art. 48, the diagonal KN coincides with the side UV of the diagonal triangle of the complete quadrangle $ABPQ$.

But, here again, this particular diagonal has no special relation to the figure, and the corresponding result must hold good of the other two diagonals. Accordingly LM lies along OU and IJ along OV.

Thus the complete quadrilateral formed by four tangents to a conic and the complete quadrangle formed by their four points of contact have the same diagonal triangle, which is self-polar for the conic.

EXAMPLES

1. Given a triangle self-polar for a conic and a point on the conic, show how to determine three other points on the conic.

2. Given a triangle self-polar for a conic and a tangent to the conic, show how to determine three other tangents.

3. Given two points A, B on a conic and the pole C of AB, construct the tangent to the conic at any given point P.

4. Given two tangents a, b to a conic and the polar c of ab, construct the point of contact of any given tangent p.

5. Prove that, of the three sides of a triangle self-polar with regard to a conic, two meet the curve in real points and one does not.

51. Graphical constructions for pole and polar. The results of the last Article enable us to obtain graphical constructions for pole and polar.

To find the polar of O, draw any two chords through O meeting the conic at real points A, B, P, Q (Fig. 20). Construct the other diagonal points U, V of the complete quadrangle $ABPQ$. UV is the required polar.

To find the pole of a straight line CR (Fig. 20), take on CR any two points K, N lying outside the conic. Draw from K and N pairs of tangents to the conic a, b and p, q respectively. Construct the other diagonals LM, IJ of the complete quadrilateral $abpq$. The intersection of LM, IJ is the required pole.

52. Conjugate ranges and pencils. Let U (Fig. 21) be any point on a given line c, whose pole is C. Then by Art. 49 the polar u of U passes through C. Hence the polars of the range $[U]$ form a pencil $[u]$. Conversely the poles of a pencil $[u]$ form a range $[U]$.

Let now u meet any other fixed straight line d at a point V. Then $[u]$ determines on d a range $[V]$. Also since the polar of U passes through V, (U, V) are conjugate points with regard to the conic. When U is known, V is uniquely determined and conversely. The ranges $[U]$ $[V]$ are termed **conjugate ranges** with respect to the conic.

The above construction clearly fails if the line d passes through C, in which case c, d are conjugate lines, and the pole D of d lies on c. It is therefore impossible to have conjugate ranges on conjugate lines, for one point in each line is then conjugate to every point in the other line.

Similarly, if the point U be joined to any other fixed point E by a line w, w passes through the pole of u, so that w, u are conjugate

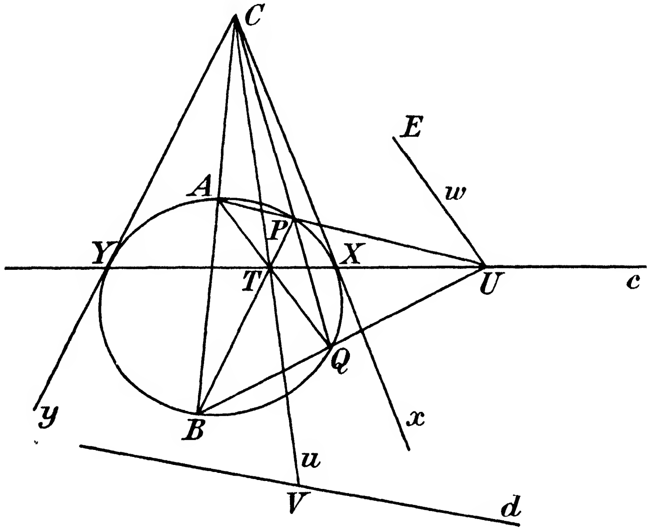


FIG. 21.

lines with regard to the conic. The pencils $[w]$, $[u]$ are said to be **conjugate pencils** with respect to the conic.

As in the case of ranges, if the vertices C, E of the pencils $[u]$, $[w]$ are conjugate points, no conjugate pencils can be obtained with C, E as vertices. For the polar c of C then passes through E. This polar is conjugate to every line u, and conversely the polar of E, which passes through C, is conjugate to every line w.

We will now prove : (1) that a range is equi-anharmonic with its polar pencil; (2) that conjugate ranges are projective; (3) that conjugate pencils are projective.

In Fig. 21, let A be any fixed point on the conic. Join AC, meeting the conic again at B. Join UA, meeting the conic again at P, and CP, meeting the conic again at Q. Then, by Art. 50, since $ABPQ$ is a quadrangle inscribed in the conic, the meet of AP, BQ lies on the polar c of C. Therefore BQ passes through the meet U of c and AP. Also the meet T of AQ, BP lies on c and CT is the side of the diagonal triangle opposite to U and is thus the polar u of U.

But, by Art. 39, A, B being fixed points, P, Q, U, T variable points,

$$A[Q] \barwedge B[Q]$$

that is
$$A[T] \barwedge B[U],$$

whence
$$[T] \barwedge [U],$$

and therefore $[U]$ is equi-anharmonic with the pencil $C[T]$, *i.e.* $[u]$, since this pencil is incident with the range $[T]$. This proves the first proposition.

Again, the range $[V]$ is incident with $[u]$. Therefore $[V]$ is equi-anharmonic with $[u]$, and therefore with $[U]$.

Hence
$$[V] \barwedge [U].$$

This proves the second proposition.

Finally, $[w] = E[U]$ is equi-anharmonic with $[U]$ and therefore with $[u]$. Hence

$$[w] \barwedge [u].$$

This proves the third proposition.

Conjugate ranges or pencils may be cobasal, *e.g.* in Fig. 21 $[U]$ and $[T]$ are clearly conjugate ranges on c and $C[U]$, $[u]$ are conjugate pencils with vertex C. If c meet the conic at X, Y, then, if U coincide with X, its polar u coincides with the tangent x at X and meets c at a point T coincident with X. Hence X is a self-corresponding point of the conjugate ranges on c. Similarly Y is a self-corresponding point of these ranges.

Again, if x, y are the tangents from C to the conic, when u coincides with x, U coincides with the point of contact X of x, and CU also coincides with x. Similarly for y. Hence x, y are self-corresponding rays of the conjugate pencils through C.

Thus conjugate ranges on a given straight line have for self-corresponding points the intersections of the line with the conic, and conjugate pencils through a given point have for self-corresponding rays the tangents from the point to the conic.

Examples

1. If through two points A and B (which are not themselves conjugate points) conjugate pencils be drawn with regard to a conic, the product of these conjugate pencils is in general a conic passing through the points of contact of the tangents from A, B to the original conic.

2. If on two lines a, b (which are not themselves conjugate lines), conjugate ranges be taken with regard to a conic, the product of these conjugate ranges is in general a conic touching the tangents to the original conic at the points where the latter is met by a, b.

3. The locus of the intersections of the polars with regard to two fixed conics of a point P lying in a given straight line is a conic.

4. If AB is a chord of a conic through a point C, T and U conjugate points on the polar of C, prove that (AT, BU) and (AU, BT) are points on the conic.

5. A, B are two fixed points on a conic, P a variable point. Prove that AP, BP meet any line conjugate to AB in conjugate points.

53. Pole and polar properties are projective. Since poles and polars involve only properties of incidence, and the harmonic relation, both of which are unaltered by projection, and since a conic projects into another conic, all the properties and constructions discussed in Arts. 48–52 are projective.

Thus pole and polar project into pole and polar, conjugate points and lines into conjugate points and lines, self-polar triangles into self-polar triangles, conjugate pencils and ranges into conjugate pencils and ranges, etc.

Results, however, which involve the line at infinity, or measurements of angles or lengths (other than cross-ratios) do not generally persist in projection.

54. Pole and polar with respect to a circle. Since the circle is a particular case of a conic, all the preceding theorems on pole and polar hold also for the circle.

Let C (Fig. 22) be the centre of a circle, P any point of the plane. From the symmetry of the circle, the points on the polar of P, which lie on chords through P equally inclined to CP, are themselves symmetrical with regard to PC. Hence the polar p of P must be perpendicular to PC.

Let it meet CP at P', and let A, B be the extremities of the diameter PC. Then, since P, P' are harmonically conjugate with regard to A, B, and C is the middle point of AB, we have, by Art. 28:

$$CP.CP' = CA^2$$

Also, since chords through C are bisected at C, the harmonic conjugates of C with regard to the extremities of such chords lie at infinity, so that the polar of C is the line at infinity c^∞.

Denote CP by r. Then its pole $R=c^\infty p$ and is the point R^∞ at infinity on p. All chords through R^∞ are parallel to p, and therefore perpendicular to r and are bisected by r. But they are clearly conjugate to r, since they pass through its pole.

In particular the diameter conjugate to r is perpendicular to r. Hence **the circle has the property that every diameter is perpendicular to its conjugate diameter.**

If any other point Q is taken on p, its polar q passes through P and is perpendicular to CQ. Hence the rays of the pencil formed by joining the points of a range to the centre of the

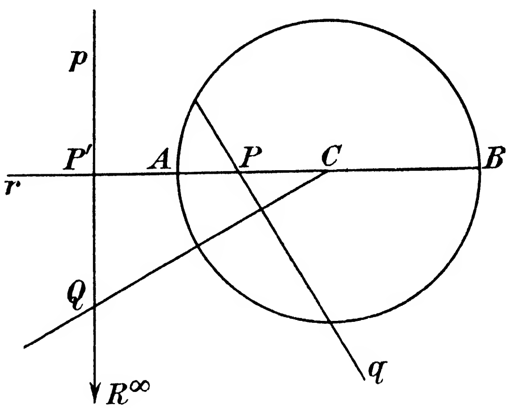


FIG. 22.

circle are perpendicular to the corresponding rays of the pencil of polars.

EXAMPLES

1. If P, Q be any two points conjugate with respect to a circle, prove that the circle on PQ as diameter is orthogonal to a given circle.

2. Show that the product of two pencils conjugate with regard to a circle, the vertex of one of which is the centre, is a circle.

3. Prove that, if two circles are orthogonal, every diameter of either which meets the other in real points is harmonically divided by that other.

4. If P and Q be points on the radical axis of two circles and P, Q be conjugate points for one circle, they are also conjugate for the other.

[The radical axis of two circles (see Art. 112) is the locus of points the tangents from which to the two circles are equal. It is the common chord if the circles cut in real points.]

5. If on a fixed line u points P, P' are taken which are conjugate with regard

to a circle, and U is the foot of the perpendicular from the centre of the circle upon u, prove that $UP.UP' = \text{constant}$.

6. In Ex. 5, if u do not meet the circle in real points, show that the constant $= -$ square of tangent from U to the circle.

55. Pole and polar with respect to a line-pair and a point-pair. If the conic be a line-pair whose components are a, b, meeting at C, the polar of any point P is the ray p through C harmonically conjugate to CP with regard to a, b. For every line through P meets CP, p and a, b in pairs of harmonic conjugates.

Conversely consider the pole of any line. Let P, Q be two points on that line. The polar of P is a line p through C and the polar of Q is a line q through C. Hence the pole of PQ is C. Thus every line not through C has C for its pole and every point not C has its polar passing through C.

If the conic be a point-pair whose components are A, B, of which c is the join, the pole of a line p is the fixed point through which pass the harmonic conjugates to p with regard to the two tangents from points of p to the conic. If Q be any point of p, QA, QB are the two tangents from Q to the point-pair. If the harmonic conjugate to p with regard to QA, QB be drawn, it meets c at a fixed point P, which is the harmonic conjugate of pc with regard to A and B. Hence the poles of every line other than c lie on c.

Also the polar of any point not on c is clearly c. For AB is the chord of contact of tangents from every such point.

56. Reciprocal polars. It appears from the previous theory that a conic s establishes a reciprocal correspondence between the elements of its plane, thus: to a point P corresponds its polar p, to a line q corresponds its pole Q. To any figure in the plane, made up of points and lines, will correspond another figure, made up of lines and points, which are the polars and poles respectively of the points and lines of the first figure with regard to the conic s, which is called the **base-conic.**

It will also be seen that properties of incidence are preserved in this reciprocal correspondence, for if P lies on q, p passes through Q; and to the meet of two lines pq corresponds the join of their poles PQ. Also from Art. 52 it follows that cross-ratio properties are unaltered. Accordingly two projective ranges will reciprocate into two projective pencils, self-corresponding points will reciprocate into self-corresponding rays, in particular perspective ranges will reciprocate into perspective pencils, four harmonic points will reciprocate into four harmonic rays ; and conversely.

Two figures ϕ_1, ϕ_2 in plane perspective reciprocate into two figures ϕ_1', ϕ_2' in plane perspective. For since collinear points reciprocate into concurrent straight lines, and conversely, therefore, if P_1P_2 passes through a fixed point U, the reciprocal lines p_1', p_2' intersect on the fixed line u' which is the reciprocal of U. Similarly, if q_1, q_2 meet on a fixed line x, they reciprocate into points Q_1', Q_2' such that $Q_1'Q_2'$ passes through the fixed point X'. Hence the figures ϕ_1', ϕ_2' are in plane perspective, the pole and axis of perspective being the reciprocals of the axis and pole of perspective, respectively, in the original figures ϕ_1, ϕ_2.

To a curve given as a locus of points will correspond a curve given as an envelope of tangents : the **degree** of one curve—that is, the number of points in which it is met by a straight line—becomes the **class** of the corresponding curve, that is, the number of tangents which can be drawn to it from any point. The join of two coincident points at P on the first curve, *i.e.* the tangent at P, reciprocates into the meet of two coincident tangents p' of the second curve, that is, the point of contact of p'.

Consider a conic obtained as the product of two projective pencils. The two projective pencils reciprocate into two projective ranges and intersections of corresponding rays into joins of corresponding points. The product of two projective pencils therefore reciprocates into the product of two projective ranges, that is, into a conic. Thus **the reciprocal of a conic is a conic.**

Let P_1 be the pole of p_1 with regard to a conic s_1, whose reciprocal with respect to a conic k is a conic s_2. From any point Q_1 of p_1, external to s_1, two tangents t_1, u_1 are drawn to s_1 ; let P_1Q_1 be denoted by q_1. Then by Art. 49 $\{p_1t_1q_1u_1\} = -1$. Now if P_2, T_2, Q_2, U_2 are the poles of p_1, t_1, q_1, u_1 with respect to k, then, since p_1, t_1, q_1, u_1 are concurrent at Q_1, the points P_2, T_2, Q_2, U_2 lie on the polar q_2 of Q_1 with respect to k, and $\{P_2T_2Q_2U_2\} = -1$. But T_2, U_2 are points of s_2, since t_1, u_1 are tangents to s_1. Hence Q_2 lies on the polar of P_2 with respect to s_2. But since q_1 passes through P_1, Q_2 lies on p_2. Thus p_2 and the polar of P_2 with respect to s_2 both coincide with the locus of Q_2 and so are identical. Hence p_2 is the polar of P_2 with respect to s_2, or **pole and polar reciprocate into polar and pole.** It follows immediately the conjugate points reciprocate into conjugate lines, and conversely; also conjugate ranges reciprocate into conjugate pencils, and conversely.

57. Principle of duality. It follows from the transformation by reciprocal polars that to every theorem concerning a figure made up of points and lines there corresponds another theorem concerning a corresponding figure made up of lines and points respectively, so that geometrical theorems appear in pairs. Several instances of this principle of duality have already been met with and it will be an instructive exercise for the student to trace such dual theorems as have already been given. As examples of such theorems we may quote the following, corresponding theorems appearing side by side :

If in two corresponding figures meets of corresponding lines lie on a fixed line, joins of corresponding points pass through a fixed point.	If in two corresponding figures joins of corresponding points pass through a fixed point, meets of corresponding lines lie on a fixed line.
The harmonic property of the complete quadrangle.	The harmonic property of the complete quadrilateral.
The meets of cross-joins of any two pairs of corresponding points of two projective ranges lie on a fixed line.	The joins of cross-meets of any two pairs of corresponding rays of two projective pencils pass through a fixed point.
The harmonic conjugates of a fixed point with regard to the two points at which any line through it meets a fixed conic lie on a fixed line.	The harmonic conjugates of a fixed line with regard to the two lines which can be drawn from any point on it to touch a fixed conic pass through a fixed point.

Reciprocal theorems are obtained at once one from the other by simply translating the language, the following being the terms interchanged :

straight line	point
join	meet
tangent to a curve	point on a curve
point of contact of a tangent	tangent at a point on the curve
lie on	pass through
range	pencil
collinear	concurrent
degree	class
locus	envelope

It should be noticed, however, that theorems true of *special* curves reciprocate into theorems true only of the curves which are the reciprocals of these special curves. Also that properties of length and angular magnitude (which are termed metrical properties) do not generally reciprocate into like properties. It will be found that the properties to which the principle of duality can be applied successfully are the *projective* properties.

58. Centre and diameters of a conic. The pole of the line at infinity with regard to a conic is called the **centre** of the conic. The centre of the conic corresponds, in the plane of the original circle, not to the centre of the circle, but to the pole of the vanishing line.

Lines through the centre of a conic are called its **diameters.** Since the centre and the point at infinity divide a diameter harmonically, a diameter is bisected at the centre.

Conjugate diameters are conjugate lines through the centre. Hence the pole of either is the point at infinity on the other. Therefore the tangents at the extremities of a diameter are parallel to its conjugate.

By the pole and polar property **chords parallel to a diameter** are divided harmonically by the point at infinity on the chords and by the conjugate diameter, that is, they **are bisected by the conjugate diameter.**

If C be the centre of a conic, P any point, the polar p of P with regard to the conic is conjugate to the diameter CP and therefore is a chord bisected by it. For if c^{∞} be the line at infinity, clearly pc^{∞}, that is, the point at infinity on p, is the pole of PC.

All diameters of an ellipse meet the curve in real points. For the vanishing line being outside the circle, its pole is inside. Every line through this pole therefore cuts the circle in real points, and the same holds good after projection.

On the other hand, of two conjugate diameters of a hyperbola, one and one only meets the curve in real points. For consider the original circle. The vanishing line c cuts the circle in real points I, J (Fig. 23a). The tangents at I, J meet at C, which is the pole of the vanishing line and is outside the circle. Of the rays through C, those which lie inside the angle ICJ meet the circle, the others do not. Now by Art. 49 any two conjugate lines through C are harmonically conjugate with regard to CI, CJ. If they meet IJ at P, P', P and P' are harmonically conjugate with regard to I and J. Hence if P' be inside IJ, P is outside and conversely, since P, P' divide IJ internally and externally in the same ratio (Art. 28); $\therefore$ if CP' cuts the circle in real points, CP does not, and conversely. Projecting IJ to infinity the property stated follows for diameters of a hyperbola.

Also note that in the hyperbola the property that CP, CP' are harmonically conjugate with regard to the tangents from C becomes:

Two conjugate diameters of a hyperbola are harmonically conjugate with regard to the asymptotes.

In the case of the parabola the vanishing line c touches the original circle at C (Fig. 23*b*), and every line CP is conjugate to c. The centre of the parabola is therefore at infinity, and its direction gives the point of contact of the line at infinity with the curve. All diameters of a parabola are parallel to this fixed direction, and are to be looked upon as conjugate to the line at infinity. The line at infinity has no definite direction, but it may be shown that to each diameter there is a definite conjugate *direction*. For let L (Fig. 23*b*) be the pole of CP, chords through L are conjugate to CP. Project c to infinity: the circle becomes a parabola, CP

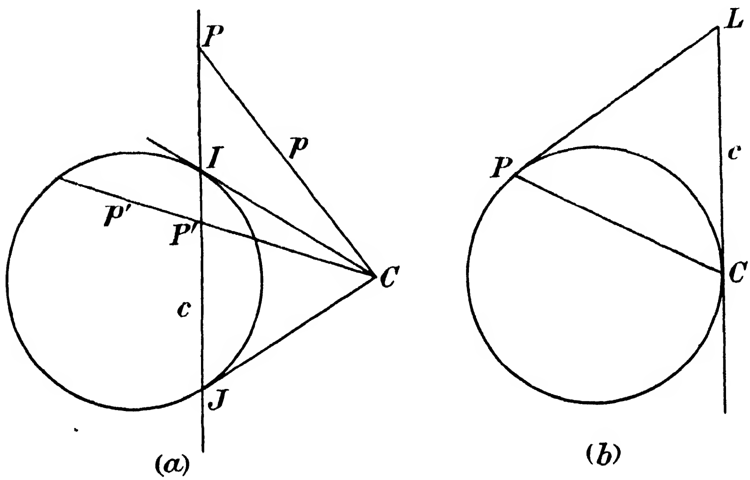


FIG. 23.

a diameter, the chords through L a system of parallel chords bisected by that diameter, PL the tangent at its extremity, which tangent is parallel to the chords. Hence **a diameter of a parabola bisects chords parallel to the tangent at its extremity.**

Because the ellipse and hyperbola have an accessible centre, they are termed **central conics.**

59. Supplemental chords of a central conic are parallel to conjugate diameters. Let AB be a diameter of a conic, P any point on the curve. Then AP, BP are termed **supplemental chords** of the conic.

Let C be the centre; thus C is the middle point of AB. Join PC meeting the conic again at Q. Then C is the middle point of

PQ. Since AB, PQ bisect one another at C, $ABPQ$ form a parallelogram. The opposite sides AP, BQ meet at the point at infinity, V^∞ on AP, and the opposite sides AQ, BP meet at the point at infinity, T^∞ on BP.

$CT^\infty U^\infty$ is therefore the diagonal triangle of the quadrangle $ABPQ$ inscribed in the conic, and so is self-polar for the conic (Art. 50). Thus CT^∞, CU^∞ are conjugate lines through the centre, that is, conjugate diameters; and they are parallel to the supplemental chords BP, AP respectively.

Conversely it is easily shown that if through the extremities A, B of any diameter, rays AP, BP are drawn parallel to a pair of conjugate diameters, their intersection P lies on the conic.

We can now show that the circle is the only conic which has more than one pair of perpendicular conjugate diameters.

For, if possible, let there be two such pairs (x_1, y_1) and (x_2, y_2). Let AB be any diameter of the conic which does not coincide with any of the four lines x_1, y_1, x_2, y_2.

By what has just been proved, if we draw, through A, AP_1 parallel to x_1 and, through B, BP_1 parallel to y_1, P_1 is a point on the conic. Similarly, if AP_1' is parallel to y_1 and BP_1' to x_1, P_1' is a point on the conic. In like manner, by drawing parallels to x_2, y_2, we obtain two other points P_2, P_2' on the conic, which therefore passes through the six points A, B, P_1, P_1', P_2, P_2'.

But since x_1, y_1 are rectangular, the angles AP_1B, $AP_1'B$ are right angles. Hence the circle on AB as diameter passes through P_1 and P_1', and similarly also passes through P_2 and P_2'. The conic and circle have therefore six points in common and must coincide altogether.

The property, that conjugate lines through the centre are at right angles, is therefore characteristic of the circle.

60. Axes of a conic. An **axis** of a conic is a diameter perpendicular to its conjugate.

Note that a conic is *symmetrical* with regard to each axis.

For if P be a point on the curve the chord through P perpendicular to an axis is bisected by that axis and therefore meets the curve again at the symmetrical point P'.

Also the axes are harmonically conjugate with regard to the asymptotes, and therefore (being perpendicular) they bisect the angles between the asymptotes by Art. 28.

A central conic, other than a circle, cannot have more than one pair of axes, by the second proposition of the last Article.

To find the position of the axes of any central conic. Let C (Fig. 24) be the centre. Draw any diameter ACB. Then, if this diameter is perpendicular to the tangents at A and B, it is perpendicular to its conjugate (Art. 58) and these give the required axes.

If ACB is not perpendicular to the tangents at A and B, describe the circle on AB as diameter. If we describe this circle in a prescribed sense (shown by the arrows in the figure) we must cross the conic at A (from the outside to the inside in Fig. 24). By the symmetry of both curves about their common centre, we must likewise cross from outside to inside at B. Hence, at some point E between A and B, we must again pass from the inside of the conic to the outside; at this point the circle and conic again intersect.

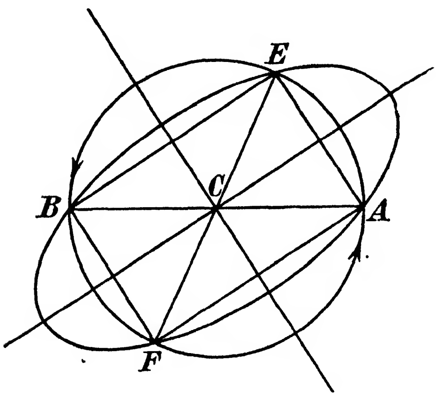


FIG. 24.

AE, BE are then perpendicular, by the property of the angle in a semicircle; and, since they are supplemental chords of the conic, the diameters parallel to them are perpendicular conjugate diameters of the conic, and therefore are the required axes.

It is easily verified that the conic and circle intersect at a fourth point F, which is diametrically opposite to E, so that $AEBF$ is a rectangle inscribed in the conic.

Thus for any central conic there exists one pair of axes, which are always real.

Both axes of an ellipse meet the curve in real points; the longer and shorter axes are called the **major** and **minor** axes of the ellipse respectively.

By Art. 58, one axis of a hyperbola meets the curve in real points. This is called the **transverse** axis. The axis which does not meet the curve in real points is called the **conjugate** axis.

In the parabola an axis is a diameter which bisects chords perpendicular to itself. Since all diameters are parallel, we have to take that one which bisects chords perpendicular to all diameters. Hence a parabola has only one axis.

The points where an axis meets a conic are called **vertices** of the curve. A parabola has only one accessible vertex.

61. Graphical construction of an ellipse when two conjugate diameters are given in position and length. Let AOB, COD be the two conjugate diameters (Fig. 25). Complete the parallelogram $EFGH$, of which they are median lines. Draw any line QRQ' parallel to the diagonal EG of this parallelogram and meeting EF, CD, GH at Q, R, Q' respectively. Join AR, BQ meeting at P. Then P is a point on the ellipse. Since QR is parallel to a fixed line, the ranges $[Q]$, $[R]$ are similar and therefore projective. Hence

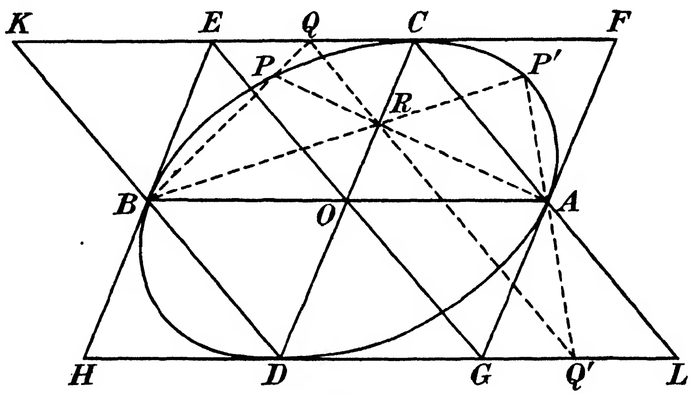


FIG. 25.

$A[R] \barwedge B[Q]$. The locus of P is therefore a conic. This conic passes through A and B, these being vertices of the two pencils. Also if QR is along EG, Q is at E, R is at O, hence AB corresponds to BE: if QR be at infinity, Q is at infinity on FE, *i.e.* on BA, R is at infinity on DC, *i.e.* on AF; hence BA corresponds to AF. The conic locus of P $\therefore$ touches EH at B, FG at A. And it passes through C, as is obvious by taking QR through C, when Q, R, P coincide at C. It has therefore five points common with the required ellipse, viz. two coincident points at A, two at B and one at C. Hence it is the required ellipse.

By taking lines QR between CA and BD the half of the ellipse inside $CEHD$ can be drawn in this way. To avoid taking distant parallels QR and to keep the construction compact, the other half of the ellipse may be drawn by joining BR, AQ' meeting at P'. P' can be shown to be a point on the ellipse by reasoning similar to that used above, and the same set of parallels can be employed to complete the ellipse.

EXAMPLES IVA

1. Without using the polar property, prove directly that if p be any fixed line in the plane of a conic, P any point on p outside the conic, t, t' the two tangents from P to the conic, the line p' harmonically conjugate to p with regard to t, t' passes through a fixed point.

2. Prove that if t be the product of conjugate pencils with regard to a conic s which have A, B for vertices, then AB has the same pole with regard to s and t.

3. Show that, if t be the product of conjugate ranges with regard to a conic s on two straight lines a, b, then the point U where a, b meet has the same polar with regard to s and t.

4. Two ranges of conjugate points with regard to a conic lie on straight lines s, s'. Show that the cross-axis of the ranges passes through the poles S, S' of the lines.

5. ABC is any triangle. A' is the pole of BC with regard to a conic, B' is the pole of CA, C' is the pole of AB. Show that the triangles ABC, $A'B'C'$ are in plane perspective.

[Use Ex. 4 noting that B, C' and B', C are pairs of conjugate points.]

6. Prove that the diagonals of a complete quadrilateral circumscribed about a conic are divided harmonically by the diagonal points of the complete quadrangle formed by their four points of contact.

7. Prove that the locus of a point which is such that its polars with regard to two given conics are perpendicular, is a conic.

8. If A, B, C, D, P are five points on a conic and $P\{ABCD\} = -1$, show that BD and AC are conjugate lines for the conic.

9. By reciprocation of Ex. 8, or otherwise, prove that, if a, b, c, d, t are five tangents to a conic and $t\{abcd\} = -1$, then bd and ac are conjugate points for the conic.

10. The tangents to a conic at A and B meet at C, and those at D and E meet at F; AB, DE meet outside the conic at O. Show that CF meets the conic at two real points X, Y, and that the diameter through O bisects XY.

11. From points on a given straight line perpendiculars are drawn to their polars with respect to a conic. Show that these perpendiculars envelope a parabola which touches the given line.

12. Prove the following construction for a parabola, given a point A, the tangent a at A, a parallel x to the axis and another point B on the curve. Construct a parallelogram $ACBD$ with AB as diagonal and BD, BC parallel to a, x respectively. Draw LM parallel to the diagonal CD to meet BC at L, BD at M. The meet of AL and a parallel to x through M is a point on the curve.

13. A, B are two fixed points in the plane of a conic s. P is a point such that the two tangents from P to s are harmonically conjugate with regard to PA, PB. Show that the locus of P is a conic.

14. State and prove the theorem obtained from Ex. 13 by reciprocation.

15. If T be any point, C the centre of a conic, N the point where the polar of T meets the diameter through T, A a point where this diameter meets the conic, prove that $CN.CT=CA^2$.

16. If a diameter of a parabola meet the curve at P and a conjugate chord at V, show that if T be the pole of that chord, T lies on the diameter and $TP=PV$.

17. Show that if a parallelogram be circumscribed to or inscribed in a conic its diagonals intersect at the centre of the conic.

18. Show that a conic is completely determined if two points and their polars and a point on the curve be given.

19. Show that a conic is completely determined if two points and their polars and a tangent to the curve be given.

20. A straight line is divided harmonically by a fixed conic s and a pair of fixed straight lines OA, OB. Prove that the envelope of the line is a conic s', which touches OA, OB at the points where they meet the polar of O with respect to s.

21. In Ex. 20 prove that the intersections of s and s' lie on two lines through O, which are harmonic conjugates, with respect to OA and OB, of the tangents from O to s.

22. Prove the following construction for a conic, given a diameter AB, a point P on the curve and the direction conjugate to AB. Complete the parallelogram $ADPE$ on AP as diagonal and whose sides AD, DP are along and conjugate to AB respectively. Let a parallel to DE meet PD at Q, PE at R. The rays AR, BQ meet on the conic.

23. Two fixed tangents OT, OT' are drawn to a conic, and the tangent at a variable point P meets them at Q, Q' and the chord of contact TT' at U. Show that $\{PQUQ'\}=-1$. What special forms does this theorem take (i) when the fixed tangents are the asymptotes of a hyperbola, (ii) when the conic is a parabola and one of the fixed tangents is the line at infinity ?

24. A pair of conjugate diameters of a given conic meet a given straight line at A and B; on AB is described a triangle APB similar to a given triangle. Prove that the locus of P is a hyperbola and find its asymptotes.

25. A given conic s touches two lines CA, CB at A, B respectively; and O is a fixed point in the plane, not belonging to s, CA, or CB. A variable line through O meets AB at X, and Y is the point of OX conjugate to X with respect to s. Prove that the locus of Y is a conic t passing through O, C, A, B, and the points of contact of the tangents to s from O.

Show also that the pole of OC with respect to t is the intersection of AB and the polar of O with respect to s.

26. Through a fixed point O a straight line is drawn to meet a fixed straight line l at P and intersects the polar of P with respect to a fixed conic s at Q. Show that as P describes the straight line l, Q describes a conic passing through three fixed points independent of the line l chosen.

Show that (1) inversion is a particular case of this construction and that (2) when O lies on s the conics corresponding to two lines l of the plane have simple contact with s at O, but three-point contact with each other.

[P, P' are said to be *inverse* points with regard to O if (O, P, P' being collinear) $OP.OP'=\text{const.}$]

EXAMPLES IVB

[The axes of co-ordinates are rectangular throughout.]

1. Two conjugate diameters of an ellipse are respectively 8″ and 6·4″ in length and the angle between them is 110° ; draw the ellipse, and measure the lengths of its principal axes.

2. Using the ruler only construct the polars of the points (1, 1) and (6, 2) with regard to the circle $x^2+y^2=9$. In any manner construct the polars of the point (3, 0) and of the points at infinity on $x=0$ and $y=x$ with regard to the same circle.

3. A conic passes through the five points (0, 0) (1, 1), (2, 1) (2·5, 0·8) (1, −2) ; using the ruler only, construct the polar of the point (1·5, 1·5) with regard to this conic.

4. Draw an isosceles right-angled triangle AOC on a hypotenuse AC of length 4 inches, and mark the middle point B of AC. With the aid of the ruler only, construct the ray through O which is conjugate to OC for every conic touching OA at A and OB at B.

Find also the pair of perpendicular conjugate rays through O.

5. Construct a line passing through the point (3, 0) and conjugate to $x=0$ with regard to the circle $(x-3)^2+(y+2)^2=1$.

6. Construct the envelope of the polar of a point P on the circle

$$x^2-4x+y^2=0$$

with regard to the circle $x^2+y^2=4$.

7. Draw the conic through the five points (0, 3) (0, 5) (1, 0) (4, 0) (2, 2) and construct its axes.

8. If the pole of perspective be (1, −2), the axis of perspective $x=-2$, construct the axis of the parabola in plane perspective with the circle $x^2+y^2-3x=0$, the vanishing line for the circle being $x=0$.

9. A, B are the points (−4, 0), (3, 0) respectively. If AR be any ray through A, P the pole of AR with regard to the circle $x^2+y^2=4$, and if BP meet AR at Q, construct the locus of Q.

10. A right circular cone of vertical angle 90° stands on a circular base k of radius 2 inches, and is cut by a plane passing through a tangent x to the circle k and making an angle of 30° with the plane of k.

Make a drawing showing, in rabatment, (i) the four vertices of the section, (ii) the point P of the section which projects into an extremity of the diameter of k parallel to x, (iii) the tangent at P, and (iv) the extremities of the diameter of the section parallel to this tangent.

CHAPTER V

NON-FOCAL PROPERTIES OF THE CONIC

62. Pascal's Theorem. If $AB'CA'BC'$ (Fig. 26) be a hexagon *
inscribed in a conic, the meets of opposite sides

$$(AB', A'B), (BC', B'C), (CA', C'A)$$

are collinear.

Let $P=(AB', A'B)$; $Q=(BC', B'C)$; $R=(CA', C'A)$; $L=(AC', BA')$; $M=(BC', CA')$.

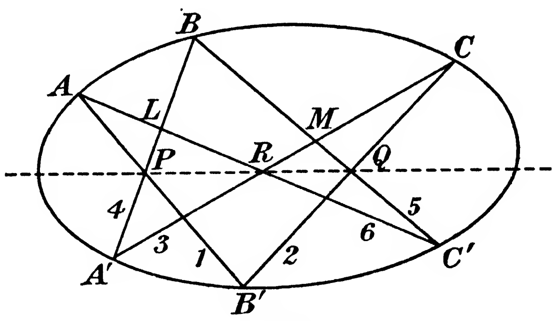


FIG. 26.

Project the four points A', B', C', B from A, C: we have by the property of the conic

$$A(A'B'C'B) \barwedge C(A'B'C'B).$$

Cutting the first pencil by $A'B$, the second by BC',

$$(A'PLB) \barwedge (MQC'B).$$

These two projective ranges have a self-corresponding point B,

* The hexagon considered here is not generally, and in graphical examples not conveniently, a *convex* figure. A similar remark applies to polygons in general, except where the contrary is distinctly stated.

$\therefore$ they are perspective and the joins of corresponding points are concurrent,

$\therefore$ $A'M$, PQ, $C'L$ are concurrent,

$\therefore$ $(A'M, C'L)$ lies on PQ, $\therefore$ R lies on PQ.

63. Brianchon's Theorem. If $ab'ca'bc'$ be a hexagon circumscribed to a conic, the joins of opposite vertices

$$p=(ab', a'b),\ q=(bc', b'c),\ r=(ca', c'a)$$

are concurrent.

This theorem is obtained immediately from Pascal's Theorem by reciprocation. The student will find it instructive to construct a proof of Brianchon's Theorem from the proof given above of Pascal's Theorem, reciprocating each step.

Pascal's and Brianchon's Theorems are conveniently expressed by the following numerical rule:

Pascal. If 1, 2, 3, 4, 5, 6 be the sides of a hexagon inscribed in a conic taken in order, then 14, 25, 36 are collinear.

The line on which they lie is called the **Pascal line** of the inscribed hexagon.

Brianchon. If 1, 2, 3, 4, 5, 6 be the vertices of a hexagon circumscribed to a conic taken in order, then 14, 25, 36 are concurrent.

The point through which they pass is called the **Brianchon point** of the circumscribed hexagon.

Examples

1. Show that, by altering the order of six points on a conic, sixty different hexagons may be formed, with sixty corresponding Pascal lines.

Show that these sixty hexagons have their Pascal lines concurrent in fours namely when they have a pair of opposite sides common.

2. Show that, in the notation of the present article for Pascal's Theorem, the lines (13, 46), (35, 62), (51, 24) are concurrent.

3. Show that (13, 46), (35, 62), (51, 24) are possible Pascal lines for the six points.

4. State and prove the results corresponding to those of Exs. 1, 2, 3 for Brianchon's Theorem.

64. Construction of conic through five points. By means of Pascal's Theorem we can construct the conic through five points.

Take the points in any convenient order, letter them in this order $AB'CA'B$. Number the *sides* $AB'=1$, $B'C=2$, $CA'=3$, $A'B=4$. Then $P=14$ in Pascal's Theorem is known. Draw any Pascal line PQR meeting 2 at Q and 3 at R. Join Q to the free end of 4, viz.

B, and R to the free end of 1, viz. A. The intersection of AR, BQ is a point C' on the conic.

By taking various Pascal lines through P we can construct any number of points on the conic.

65. Construction of conic touching five lines. Similarly let five tangents to a conic be given. Letter them in order $ab'ca'b$. Number the *vertices* $ab'=1$, $b'c=2$, $ca'=3$, $a'b=4$. Then $p=14$ is a fixed line. On p take any Brianchon point B. Let q be the join of B to 2, r the join of B to 3. q meets b the open side through 4 at the vertex 5, r meets a the open side through 1 at the vertex 6. 56 is the tangent c' to the conic.

By taking different Brianchon points on p, we can construct the conic by tangents as an envelope.

66. Coincident elements. Important particular cases of Pascal's and Brianchon's Theorems occur when two elements coincide. In this case it is important to bear in mind that if the coincident elements are points, these points have to be taken as consecutive vertices of the Pascal hexagon and the side of the hexagon joining them is to be interpreted as the corresponding tangent. If the coincident elements are tangents, these are consecutive sides of a Brianchon hexagon, and the vertex of the hexagon common to them is interpreted as the corresponding point of contact.

In all cases we shall write repeated elements twice over when considering Pascal and Brianchon hexagons, thus

$$AABCDD$$

will be considered a hexagon, and its sides taken in order are

AA (tangent at A),
AB,
BC,
CD,
DD (tangent at D),
DA.

EXAMPLES

1. Given five points A, B, C, D, E on a conic, construct the tangent to the conic at any one of them.

2. Given five tangents a, b, c, d, e to a conic, construct the point of contact of any one of them.

67. Asymptote properties of the hyperbola. Let C (Fig. 27) be the centre of a hyperbola, A^∞, B^∞ the points at infinity on the two asymptotes, P, Q two points on the curve. Consider the Pascal hexagon $A^\infty A^\infty P B^\infty B^\infty Q$; its sides taken in order are as follows: $A^\infty A^\infty = 1 =$ the asymptote CA; $A^\infty P = 2 =$ the parallel PL to CA; $PB^\infty = 3 =$ the parallel PM to CB; $B^\infty B^\infty = 4 =$ asymptote CB; $B^\infty Q = 5 =$ the parallel QL to CB; $QA^\infty = 6 =$ the parallel QM to CA.

Hence $14 = C$, $25 = L$, $36 = M$ and C, L, M are collinear, that is, if on PQ as diagonal a parallelogram be described whose sides

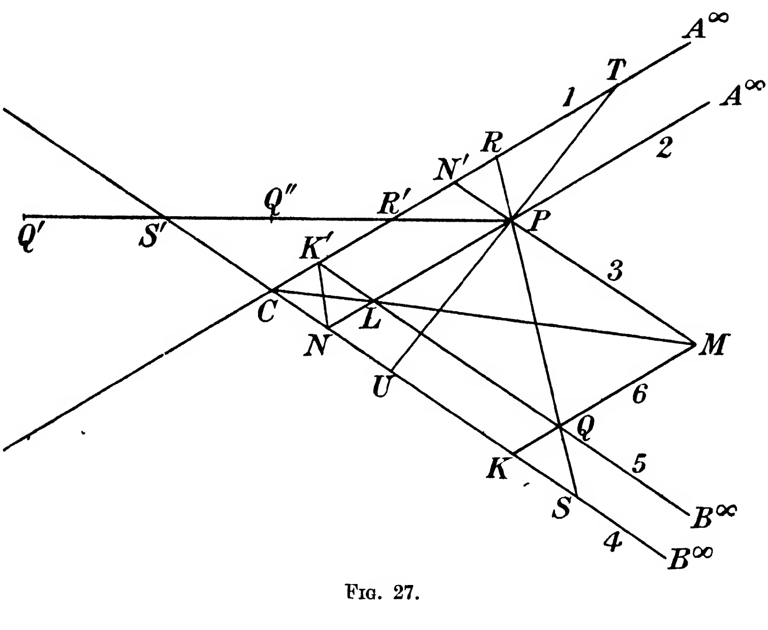


FIG. 27.

are parallel to the asymptotes the other diagonal passes through the centre.

It follows that the parallelograms $PNCN'$, $QKCK'$, being made up of $LK'CN$ and of the complements $LPN'K'$, $KQLN$ respectively, are equal in area. Hence **if through a point P on a hyperbola parallels are drawn to the asymptotes, the parallelogram thus formed is of constant area.**

Also $K'N$ is parallel to PQ, for the parallelograms $CNLK'$, $LQMP$ are clearly similar and similarly placed. Hence if PQ meet the asymptotes at R, S, $PR = NK' = QS$ (opposite sides of parallelograms). Hence **the distances intercepted on any straight line between the curve and the asymptotes are equal.**

This last property furnishes an easy method for drawing a hyperbola when the asymptotes and one point P on the curve are given. Draw a variable ray through P meeting the asymptotes at R and S. On this ray take a point Q such that $SQ = PR$. Q describes the hyperbola.

Care must be taken that in all cases PQ and RS shall have the *same mid-point*. Thus in Fig. 27, when the ray is drawn as $PR'S'$, Q must be taken at Q' outside $S'R'$ and *not* at Q'' inside.

If the points P, Q coincide the property last proved becomes: **the intercept of a tangent to a hyperbola between the asymptotes is bisected at the point of contact.**

If TU (Fig. 27) be drawn through P parallel to NN', $TP = NN' = PU$. Hence TU is the tangent at P.

Also the triangle TCU = twice parallelogram $PNCN'$

= const. by property proved above.

Hence **a variable tangent to a hyperbola cuts off from the asymptotes a triangle of constant area.**

EXAMPLES

1. Deduce the results of the above Article from the property that a pair of conjugate diameters of a hyperbola are harmonically conjugate with regard to the asymptotes, without using Pascal's Theorem.

2. Obtain the theorem that a variable tangent to a hyperbola cuts off from the asymptotes a triangle of constant area by applying Brianchon's Theorem to the hexagon $aapbbq$, a, b being the asymptotes, p, q any two tangents.

68. Construction of a hyperbola, given three points and the direction of both asymptotes. We first of all proceed to construct the centre.

If A, B, C be the three given points, construct the parallelograms on AB, BC as diagonals whose sides are parallel to the asymptotes. The centre is then the intersection of the other two diagonals (Art. 67). The asymptotes are now known in position and the hyperbola may be constructed by the method of Art. 67.

69. Given four points on a hyperbola and the direction of one asymptote, to construct the direction of the other asymptote. Let A, B, C, D be the four points; let E^∞ be the direction of the given asymptote, F^∞ that of the required asymptote. Then, considering the hexagon $ABCDE^\infty F^\infty$, the points P = intersection of AB, DE^∞, Q^∞ = intersection of BC and line at infinity, R = intersection of CD, $F^\infty A$ are collinear. Hence if the parallel through P to BC meet CD at R, AR gives the direction required. We can now use the method of Art. 68 to construct the asymptotes in position and hence to draw the hyperbola; or, considering the hexagon $ABCDE^{\infty}E^{\infty}$, the points $P_1=(AB,\ DE^{\infty})$, $Q_1=$ intersection of BC and the asymptote through E^{∞}, $R_1=(CD,\ E^{\infty}A)$ are collinear. Hence P_1R_1, which is known, meets BC at Q_1 and the line Q_1E^{∞} is one asymptote. The asymptote through F^{∞} is similarly constructed.

70. Parabola from four tangents. Since the line at infinity i^{∞} is a tangent to the parabola, four tangents a, b, c, d define the

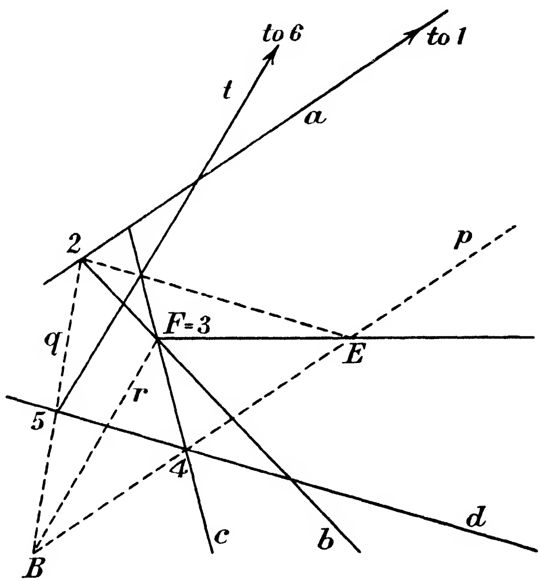


FIG. 28.

curve. Let t be any required tangent. Consider the Brianchon hexagon $i^{\infty}abcdt$ (Fig. 28). Let 1, 2, 3, 4, 5, 6 be the vertices $i^{\infty}a$, ab, bc, cd, dt, ti^{∞} in order. p or 14 is then the parallel through cd to a. On this take any Brianchon point B. Join $2B$ meeting d at 5: the parallel through 5 to $3B$ is the tangent required.

By this method we can draw a tangent to a parabola parallel to any required direction. For draw through 3 a parallel r to this direction to meet p at B; B is the corresponding Brianchon point.

Also we can construct at once the direction of the axis. For we have to find the point of contact of the line at infinity. To do this consider the Brianchon hexagon $abcdi^\infty i^\infty$. We have (ab, di^∞) $(bc, i^\infty i^\infty)$ $(cd, i^\infty a)$ are concurrent. Hence if the parallels through cd to a and through ab to d meet at E and bc is F, EF goes through the point of contact of i^∞, that is, it is parallel to the axis.

The tangent perpendicular to the axis is then constructed.

The point of contact of a tangent t is readily found when we know two other tangents and the direction of the axis. For consider the hexagon $ttai^\infty i^\infty b$; $(tt, i^\infty i^\infty)$ (ta, bi^∞) (ai^∞, bt) are concurrent. Hence through the meets of the tangent t with each

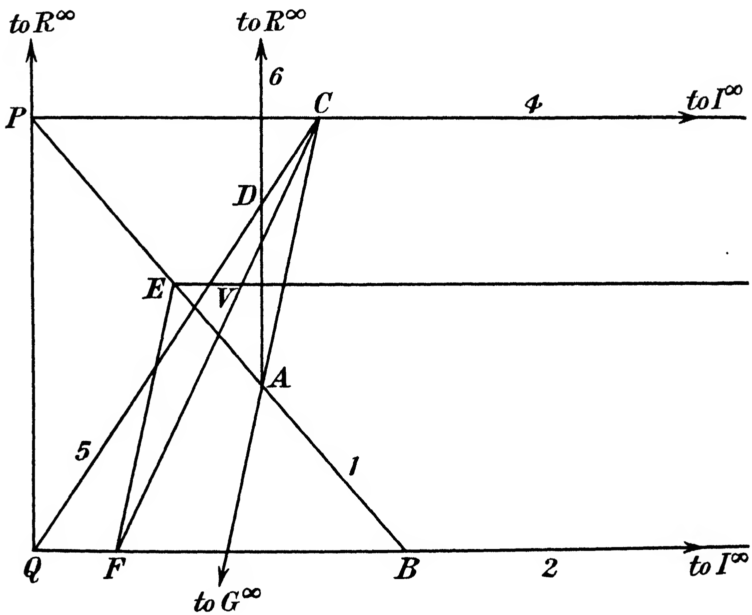


FIG. 29.

of the given tangents draw a parallel to the other tangent. The line drawn parallel to the axis through the intersection of these parallels meets the tangent t at its point of contact.

Construct therefore the point of contact of the tangent perpendicular to the axis. This is the vertex of the parabola. The line through the vertex in the direction of the axis is the axis.

71. Parabola from three points and direction of axis. Let three points A, B, C be given and the point I^∞ on the axis, *i.e.* the direction of the axis. We first construct a point D on the line through A perpendicular to the axis (Fig. 29). Considering the

Pascal hexagon $ABI^\infty I^\infty CD$ we have $(AB, I^\infty C)$, (BI^∞, CD), $(I^\infty I^\infty, DA)$ are collinear. But $I^\infty I^\infty$ is the tangent at I^∞ to the parabola and is therefore the line at infinity. Thus if P is the meet of AB and the parallel to the axis through C, Q the meet of CD and the parallel to the axis through B, R^∞ the point at infinity on DA, then P, Q, R^∞ are collinear or PQ is parallel to DA. Now P is fixed, A, B, C being given; PQ being perpendicular to the axis, Q is found and CQ meets the perpendicular to the axis through A at the point D required. The line bisecting AD at right angles is therefore the axis of the parabola.

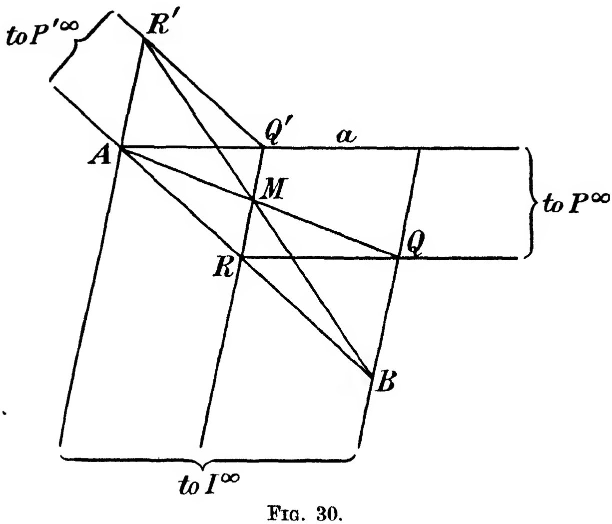


FIG. 30.

Let V be the vertex. Consider the Pascal hexagon

$$ABI^\infty I^\infty VC.$$

Then $$(AB, VI^\infty), (BI^\infty, VC), (I^\infty I^\infty, CA)$$

are collinear. Let AB meet the axis at E, CA meet the line at infinity at G^∞, VC meet the parallel to the axis through B at F. Then F lies on the parallel to CA through E and V lies on FC. Thus V is known.

72. Parabola from a tangent and its point of contact, another point and the direction of the axis. Let a, A represent the tangent and its point of contact (Fig. 30), B the other point, I^∞ the point at infinity on the axis, M any other point on the curve.

Two constructions may be used, according as we prefer to describe the curve by rays through A or by rays through B.

In the first construction draw lines QR parallel to the tangent at A to meet the parallel to the axis through B and AB at Q, R respectively. Then AQ meets the parallel to the axis through R at a point M on the curve. The result follows by considering the hexagon $AAMI^{\infty}I^{\infty}B$; 14 is P^{∞}, the point at infinity on the tangent at A: $P^{\infty}QR$ is then the Pascal line.

In the second construction draw a parallel to AB to meet AB at P'^{∞}, the tangent at A at Q' and the parallel to the axis through A at R'. Join BR' meeting the parallel to the axis through Q' at a point M. Then M is on the parabola. For $P'^{\infty}Q'R'$ is the Pascal line of the hexagon $BAAI^{\infty}I^{\infty}M$.

Example

Prove that, in Fig. 30, MQ' is proportional to $(AQ')^2$.

73. Inscribed and circumscribed triangles. Let ABC be a triangle inscribed in a conic.

From the Pascal hexagon $AABBCC$ we find

$$(AA,\ BC),\ (AB,\ CC),\ (BB,\ CA)$$

are collinear, or the sides of an inscribed triangle meet the tangents at the opposite vertices at collinear points.

If abc be a triangle circumscribed about a conic, it follows in like manner from the Brianchon hexagon $aabbcc$ that the joins of the vertices to the points of contact of the opposite sides are concurrent.

Example

The sides BC, CA, AB of a triangle touch a conic at the points P, Q, R respectively. Show that

$$BP.CQ.AR = PC.QA.RB.$$

74. Carnot's Theorem. If the sides BC, CA, AB of a triangle ABC meet a conic at P, P'; Q, Q'; R, R' respectively, then

$$\frac{BP.BP'}{CP.CP'} \cdot \frac{CQ.CQ'}{AQ.AQ'} \cdot \frac{AR.AR'}{BR.BR'} = 1.$$

Since every conic is obtained from a circle by projection (Art. 33) and the triangle ratios $\dfrac{BP}{CP} \cdot \dfrac{CQ}{AQ} \cdot \dfrac{AR}{BR}$ and $\dfrac{BP'}{CP'} \cdot \dfrac{CQ'}{AQ'} \cdot \dfrac{AR'}{BR'}$ are unaltered by projection (Art. 31), it will be sufficient to prove the above theorem for a circle.

But since the product of segments of chords of a circle through a given point is constant, we have in this case

$$AQ.AQ' = AR.AR'\,; \quad BR.BR' = BP.BP'\,;$$
$$CP.CP' = CQ.CQ',$$

whence the result required follows immediately.

Examples

1. Prove the converse of Carnot's Theorem, namely that if on the sides BC, CA, AB respectively of a triangle ABC, points P, P'; Q, Q'; R, R' are taken, such that

$$\frac{BP.BP'}{CP.CP'} \cdot \frac{CQ.CQ'}{AQ.AQ'} \cdot \frac{AR.AR'}{BR.BR'} = 1,$$

then the six points lie on a conic.

2. ABC is a triangle. A conic s meets BC at P_1, P_2, CA at Q_1, Q_2, AB at R_1, R_2. P_1', P_2' are harmonically conjugate to P_1, P_2 with respect to B and C; Q_1', Q_2' are harmonically conjugate to Q_1, Q_2 with respect to C and A; R_1', R_2' are harmonically conjugate to R_1, R_2 with respect to A and B. Prove that P_1', P_2', Q_1', Q_2', R_1', R_2' lie on a conic.

3. If from two points A, B pairs of tangents (AP, AP'), (BQ, BQ') be drawn to a circle centre O, prove that

$$\frac{\sin BAP}{\sin ABQ} \cdot \frac{\sin BAP'}{\sin ABQ'} = \frac{OB^2}{OA^2}.$$

4. Using the result of Ex. 3, prove that, if AP, AP'; BQ, BQ'; CR, CR' be tangents to a conic from three points A, B, C,

$$\frac{\sin BAP.\sin BAP'}{\sin CAP.\sin CAP'} \cdot \frac{\sin CBQ.\sin CBQ'}{\sin ABQ.\sin ABQ'} \cdot \frac{\sin ACR.\sin ACR'}{\sin BCR.\sin BCR'} = 1,$$

and, conversely, that if the above relation is satisfied, the six lines touch a conic.

5. ABC is a triangle. From A, B, C pairs of tangents p_1, p_2; q_1, q_2; r_1, r_2 are drawn to a conic s. p_1', p_2' are harmonically conjugate to p_1, p_2 with regard to AB, AC; with a similar notation for q_1', q_2'; r_1', r_2'. Prove that the six lines p_1', p_2', q_1', q_2', r_1', r_2' touch a conic.

75. Newton's Theorem on the product of segments of chords of a conic. If PP', QQ' (Fig. 31) be two chords of a conic, intersecting at O, and RR', SS' be two other chords intersecting at V and parallel to PP', QQ' respectively, then

$$\frac{OP.OP'}{OQ.OQ'} = \frac{VR.VR'}{VS.VS'}.$$

Let SS' and PP' meet at U and let J^∞ be the meet of QQ' and SS'.

Applying Carnot's Theorem to the triangle OUJ^∞, we have

$$\frac{OP.OP'}{UP.UP'} \cdot \frac{US.US'}{J^\infty S.J^\infty S'} \cdot \frac{J^\infty Q.J^\infty Q'}{OQ.OQ'} = 1.$$

But, since Q, Q', S, S', are accessible points,

$$J^{\infty}Q : J^{\infty}S = 1 \text{ and } J^{\infty}Q' : J^{\infty}S' = 1.$$

Thus
$$\frac{OP.OP'}{OQ.OQ'} = \frac{UP.UP'}{US.US'}.$$

Considering similarly the triangle formed by PP', RR', SS', we find

$$\frac{UP.UP'}{US.US'} = \frac{VR.VR'}{VS.VS'}.$$

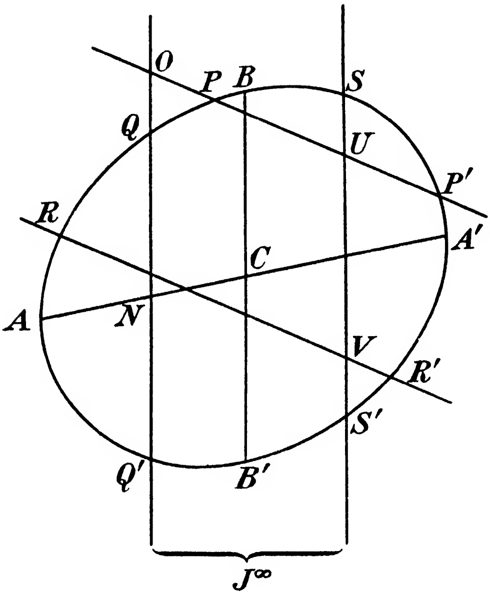


FIG. 31.

Combining the last two results, we obtain the theorem originally stated. Thus the ratio of products of segments of chords drawn from a point O in given directions is independent of the position of O.

If we take V at the centre C the parallel chords are bisected at the centre.

Hence **the ratio of products of segments of chords of a conic through any point is equal to the ratio of the squares of the parallel semi-diameters.**

If $Q'=Q$, $P'=P$ we have: **two tangents to a conic from any point are in the ratio of the parallel semi-diameters.**

76. Oblique ordinate and abscissa referred to conjugate diameters. If the chord PP' (Fig. 31) coincide with the diameter AA' conjugate to the chord QQ' the property of the last article takes the form

$$\frac{NQ.NQ'}{NA.NA'}=\text{const.}=\frac{CB^2}{CA^2},$$

CB being the semi-diameter conjugate to CA; or since

$$NQ'=-NQ,$$

$$\frac{QN^2}{AN.NA'}=\text{const.}=\frac{CB^2}{CA^2}.$$

In Fig. 31 the conic is an ellipse, and the diameter BCB' meets the curve in real points. Therefore CB^2 is positive and $AN.NA'$ is positive, so that N lies between A and A'.

But if the conic be a hyperbola we know that if ACA' meet the curve in real points, its conjugate BCB' does not meet the curve. Hence there is no real semi-diameter CB.

Nevertheless the theorem of Art. 75 holds good and

$$\frac{QN^2}{AN.NA'}=\text{a constant},$$

but N is outside AA' and the constant is negative. If we then construct a length CB_1 such that

$$-\frac{CB_1{}^2}{CA^2}=\frac{QN^2}{AN.NA'},$$

and lay it off along the diameter conjugate to AA', CB_1 may be spoken of as the **absolute length,** or simply the length, of the semi-diameter conjugate to CA.* But it should be carefully remembered that B_1 is *not* a point on the hyperbola.

Returning to the relation

$$\frac{QN^2}{AN.NA'}=\frac{CB^2}{CA^2}$$

* The term absolute length will also be used, in the case of a *real* segment PQ, to denote $|PQ|$, or the numerical length without regard to sign.

for the ellipse, we have

$$\frac{QN^2}{CB^2}=\frac{(AC+CN)(NC+CA')}{CA^2}$$

$$=\frac{-(CN-CA)(CA+CN)}{CA^2}$$

$$=\frac{CA^2-CN^2}{CA^2},$$

or
$$\frac{CN^2}{CA^2}+\frac{QN^2}{CB^2}=1.$$

For the hyperbola

$$\frac{QN^2}{CB_1{}^2}=-\frac{CA^2-CN^2}{CA^2},$$

or
$$\frac{CN^2}{CA^2} \quad \frac{QN^2}{CB_1{}^2}=1.$$

In the case of the parabola A' is at infinity.

Take two chords Q_1Q_1', Q_2, Q_2' conjugate to the same diameter,

$$\frac{Q_1N_1{}^2}{AN_1.N_1A'}=\frac{Q_2N_2{}^2}{AN_2.N_2A'},$$

$$\frac{Q_1N_1{}^2}{AN_1}=\frac{Q_2N_2{}^2}{AN_2}\cdot\frac{N_1A'}{N_2A'};$$

now since A' is at infinity $N_1A' : N_2A'=1$.

Hence
$$\frac{Q_1N_1{}^2}{AN_1}=\frac{Q_2N_2{}^2}{AN_2},$$

or
$$\frac{QN^2}{AN}=\text{constant}.$$

This constant is known as the **parameter** of the chords conjugate to the given diameter.

The above relations lead to the well-known analytical equations of the ellipse, hyperbola and parabola referred to conjugate directions.

77. Intersections of a conic and a circle. Let a circle meet a conic at four points P, Q, R, S, and let PQ meet RS at O.

Then if CL, CM are the semi-diameters of the conic parallel to PQ, RS we have

$$\frac{CL^2}{CM^2}=\frac{OP.OQ}{OR.OS}=1.$$

by the property of segments of chords of a circle.

Hence the semi-diameters CL, CM are equal. Now the extremities of all equal semi-diameters lie on a circle concentric with the conic. Thus there can be only four of them and, by the symmetry of the conic with regard to the axes, they lie pair and pair on two diameters equally inclined to the axes. Thus CL, CM are equally inclined to the axes, and the common chords PQ, RS of the conic and circle are equally inclined to the axes.

The same holds of the other pairs of opposite common chords, viz. PS, RQ ; PR, QS.

In particular, if a circle and conic have three coincident intersections P, Q, R, the common chord PS and the common tangent at P are equally inclined to the axes.

This property enables us to construct graphically the circle of curvature at P to a given conic.

If P approaches the extremity of an axis, the common tangent at P becomes parallel to the other axis, and PS approaches the tangent at P. Thus S approaches P and the circle and conic have four coincident intersections at P. Accordingly the circle of curvature at the extremity of an axis has four-point contact with the conic.

Examples

1. Show how to find graphically the directions of the axes of any given conic without first finding the centre.

2. Show that, if a circle touch a conic at A and cut it again at B and C, the common chord BC and the common tangent at A are equally inclined to the axes.

3. If a circle and conic have double contact, the common chord of contact is parallel to an axis.

78. Every ellipse can be derived from a circle by an orthogonal projection. For consider the orthogonal projection of a circle upon any plane through a diameter x. A line perpendicular to this axis of perspective x is still perpendicular to x after projection and rabatment about x into the original plane, and if P be any point of the original figure, PX the perpendicular from P on x, P' the corresponding point of the rabatted projection, P' lies on PX and $P'X : PX =$ cosine of dihedral angle θ between the two planes.

Thus the pole of perspective is at infinity in the direction perpendicular to x. The perspective relation between the circle and ellipse figures is equivalent to a stretch (*cf.* Art. 13), the stretch-ratio being $\cos\theta$.

Also because the projection is cylindrical the lines at infinity correspond: therefore their poles, that is, the centres, correspond. It follows that conjugate diameters of the circle project into conjugate diameters of the ellipse: in particular x and the perpendicular diameter of the circle, being conjugate diameters, project into perpendicular conjugate diameters of the ellipse, since a perpendicular to x remains perpendicular to x. These give the axes of the ellipse. If a be the radius of the circle, a and $a\cos\theta$ are the major and minor semi-axes of the ellipse.

Conversely an ellipse of semi-axes a and b $(a > b)$ can be obtained in this manner by projecting a circle of radius a orthogonally upon a plane making with the plane of the circle an angle

$$\theta = \cos^{-1}\left(\frac{b}{a}\right).$$

An ellipse being completely given by its principal axes (*see* Art. 61), it follows that every ellipse can be obtained in this way from a circle on its major axis as diameter.

The circle on the major axis of an ellipse as diameter is called its **auxiliary circle.** Thus **the ellipse and its auxiliary circle are derivable one from the other by a stretch parallel to the minor axis.**

EXAMPLES

1. Show that, if P, P' be points where a perpendicular to the major axis of an ellipse meets the curve and the auxiliary circle respectively, the tangents at P, P' meet on the major axis.

2. Prove that an ellipse can be obtained from the circle on its minor axis as diameter by a stretch parallel to the major axis.

79. The conjugate parallelogram. A **conjugate parallelogram** is one whose sides are the tangents at the extremities of two conjugate diameters. Clearly no real conjugate parallelogram can exist, except in the case of the ellipse.

Consider the ellipse as defined by the stretch of its auxiliary circle. Since by Art. 78 conjugate diameters correspond to conjugate diameters and parallel lines to parallel lines (because the vanishing lines are at infinity) it follows that a conjugate parallelogram for the ellipse corresponds to a conjugate parallelogram for the circle.

But a conjugate parallelogram for a circle is a circumscribed square, because conjugate diameters of a circle are at right angles.

Now, in any stretch, corresponding areas are to one another in the stretch ratio. For consider an elementary parallelogram $PQRS$ (Fig. 32) of which the sides PQ, RS are parallel to the stretch axis and the sides PS, QR are parallel to the direction of stretch. Let these meet the stretch axis at X, Y respectively. $PQRS$ transforms into a parallelogram $P'Q'R'S'$ in which $P'Q'$, $R'S'$ are parallel to the stretch axis. For if λ be the stretch ratio

$$P'X = \lambda . PX = \lambda . QY = Q'Y \quad \text{and} \quad S'Y = R'Y = \lambda . SZ.$$

Hence $$P'S' = \lambda(XS - XP) = \lambda . PS$$

and parallelogram $P'Q'R'S'$: parallelogram $PQRS = S'P' : SP = \lambda$.

Breaking up any area into such elementary parallelograms and adding we see that $\lambda =$ ratio of two corresponding areas.

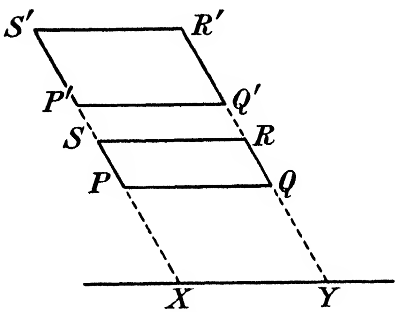


Fig. 32.

Hence area of any conjugate parallelogram of the ellipse

$$= \frac{b}{a} \times \text{area of corresponding circumscribed square of the auxiliary circle}$$

$$= \frac{b}{a} . 4a^2 = 4ab.$$

Thus **the conjugate parallelogram of an ellipse is of constant area.** Calling p the perpendicular from the centre on any given tangent, d the length of the semi-diameter parallel to this tangent, the area of the conjugate parallelogram of which the given tangent is a side is clearly $4pd$, for $2d$ is the base of the parallelogram and $2p$ is the height. $\therefore$ $4pd = 4ab$, *i.e.* $pd = ab$.

Examples

1. Show that the diagonals of a conjugate parallelogram are themselves conjugate diameters.

2. Show how to construct geometrically the equal conjugate diameters of an ellipse, given its axes.

3. Show that the area of the ellipse is πab.

80. Sum of squares of two conjugate semi-diameters. Let CP', CQ' (Fig. 33) be two perpendicular semi-diameters of the auxiliary circle, CP, CQ the corresponding diameters of the ellipse.

Then CP, CQ are conjugate. Also $P'PM$, $Q'QN$ are perpendicular to the major axis.

Then from the stretch property, if ϕ be the angle ACP',

$$PM = \frac{b}{a} P'M = \frac{b}{a} a \sin \phi = b \sin \phi ;$$

$$QN = \frac{b}{a} Q'N = \frac{b}{a} a \cos \phi = b \cos \phi.$$

$$CM = a \cos \phi ; \quad CN = a \sin \phi ;$$

$$\begin{aligned} CP^2 + CQ^2 &= CM^2 + PM^2 + QN^2 + CN^2 \\ &= a^2 \cos^2 \phi + b^2 \sin^2 \phi + b^2 \cos^2 \phi + a^2 \sin^2 \phi \\ &= a^2 + b^2. \end{aligned}$$

Hence **the sum of the squares of two conjugate diameters of an ellipse is constant.**

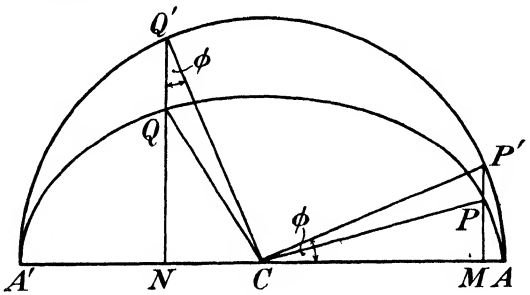


Fig. 33.

81. Pseudo-conjugate parallelogram. Let ACA' (Fig. 34) be a diameter of a hyperbola meeting the curve at real points A, A'. Let the tangent at A meet the asymptotes at D, E and the tangent at A' meet them at F, G. Then, because the tangents at A, A' are parallel, $DEFG$ is a parallelogram of which the asymptotes are diagonals.

If P be a point on the curve and PN the chord through P conjugate to ACA' meeting ACA' at N,

$$\frac{PN^2}{AN.NA'} = -\frac{CB_1{}^2}{CA^2},$$

CB_1 being the absolute length (*see* Art. 76) of the diameter conjugate to ACA'.

But $$\frac{PN^2}{AN.NA'} = -\left(\frac{AN}{A'N}\right)\left(\frac{PN}{AN}\right)^2.$$

If P moves off to infinity on the hyperbola in the direction of the asymptote CD, AP becomes parallel to CD. The triangles CAD, ANP become similar, and $\left(\frac{PN}{AN}\right)^2$ becomes equal to $\left(\frac{AD}{CA}\right)^2$; also $\frac{AN}{A'N}$ approaches unity.

Hence
$$\frac{CB_1^2}{CA^2} = \frac{AD^2}{CA^2},$$
or
$$CB_1 = AD.$$

Thus the intercept of a tangent between the asymptotes measures

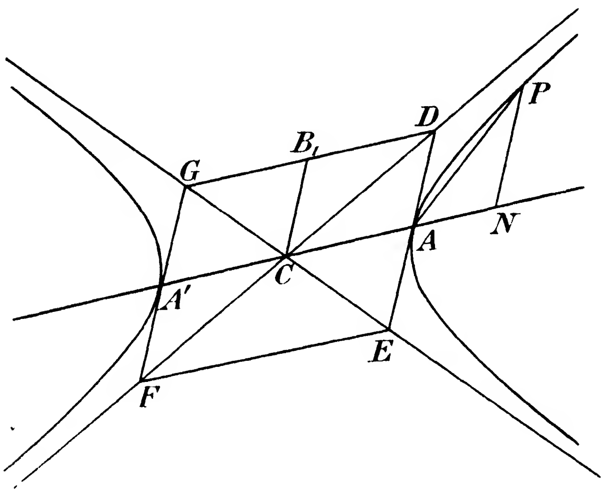


FIG. 34.

the absolute length of the parallel diameter. The parallelogram $DEFG$ is therefore a **pseudo-conjugate** parallelogram. Its median lines are conjugate diameters, but only one pair of sides touches the curve.

We have seen (Art. 67) that the area of the triangle ECD cut off from the asymptotes by any tangent to the curve is constant. **The area of the pseudo-conjugate parallelogram** $DEFG$ is four times the area of the triangle ECD and **is therefore also constant.**

If p = perpendicular from C on tangent at A, $d = CB_1$ = absolute length of diameter conjugate to CA, $DE = 2d$, and area of triangle $ECD = \frac{1}{2}p \cdot 2d = pd$. Hence **in the hyperbola as in the ellipse** $pd =$ **constant.** Taking the case where the sides of the pseudo-conjugate parallelogram are parallel to the axes, the constant $= ab$ where a is the semi-transverse axis and b is the absolute length of the conjugate semi-axis.

Further, in Fig. 34, the area of the triangle GCD is equal to that of the triangle CDE, and therefore constant. Hence GD touches at its middle point B_1 a second hyperbola having the same asymptotes. This second hyperbola, which is also the locus of B_1, is termed the **conjugate hyperbola** of the first one; the transverse and conjugate axes are interchanged, in position and absolute length, when we pass from a hyperbola to its conjugate hyperbola.

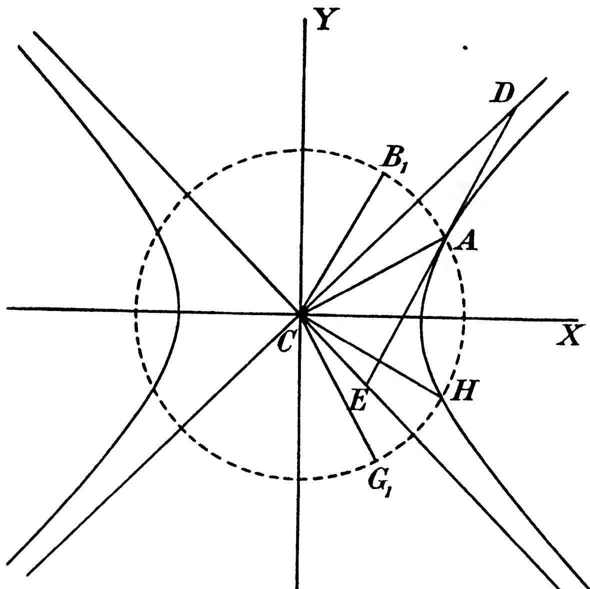


FIG. 35.

82. Difference of squares of absolute lengths of conjugate semi-diameters of a hyperbola. From the triangle CED (Fig. 35) we have, since CA is the median,

$$CD^2 + CE^2 = 2(CA^2 + AD^2)$$

by a well-known relation.

Also $$DE^2 = CE^2 + CD^2 - 2CE.CD \cos ECD,$$

that is, $$4AD^2 = 2CA^2 + 2AD^2 - 2CE.CD \cos ECD,$$

or $$CA^2 - AD^2 = CE.CD \cos ECD.$$

But $CE.CD$ is constant since the area of the triangle ECD is constant. Hence, the angle ECD being likewise constant,

$$CA^2 - AD^2 = \text{constant}.$$

The constant is easily seen to be $a^2 - b^2$ by taking CA, AD parallel to the transverse and conjugate axes.

83. Rectangular hyperbola. If the angle between the asymptotes is a right angle the curve is called a **rectangular hyperbola.**

Conjugate diameters of a rectangular hyperbola are equal in absolute length. For if DE (Fig. 35) be the tangent at A to such a hyperbola, meeting the asymptotes at D and E; since the angle at C is a right angle, the circle on DE as diameter passes through C. Hence $AC = AD$. Also CA, AD, *i.e.* CA, CB_1 are then equally inclined to the asymptotes (CB_1 having the same meaning as in Art. 81).

In particular the transverse and conjugate semi-axes are equal for a rectangular hyperbola.

Again consider the semi-diameter CH perpendicular to CB_1. Because CE, CH are perpendicular to CD, CB_1 respectively, the angle ECH = angle DCB_1 = angle ACD. CA, CH are thus equally inclined to the axes CX, CY and CH is therefore real and equal to CA, that is to CB_1. Thus the absolute lengths of perpendicular semi-diameters of a rectangular hyperbola are equal.

The diameter CG_1 conjugate to CH makes the angle G_1CE = angle ECH = angle ACD. It is therefore perpendicular to CA. We have thus a set of four diameters equal in absolute length.

Notice that this does not invalidate the result mentioned in Art. 77 that only one diameter exists equal to a given diameter; and that this diameter is equally inclined to the axes with the given diameter. For the lengths CB_1, CG_1 are not semi-diameters at all, but merely the analogues of semi-diameters: they are only called such by a convention, B_1, G_1 not being points on the curve.

84. Radius of curvature at a point on a conic. Since a conic and its circle of curvature at P may be regarded as having three-line contact along the tangent at P, they may be brought into plane perspective by taking this tangent as the axis x of perspective, and a certain point O on x as pole of perspective (Art. 47).

Denote the points belonging to the circle figure by the suffix 1, and those belonging to the conic figure by the suffix 2.

Let K_1 (Fig. 36) be the centre of the circle, Q_1 the other extremity of the diameter of the circle through P, R_1 an extremity of the perpendicular diameter of the circle.

Since chords of the circle parallel to x transform into chords of the conic parallel to x, and conjugate lines into conjugate lines, and since PQ_1 is conjugate to all chords of the circle parallel to x, PQ_2 is conjugate to chords of the conic parallel to x. It is accordingly a diameter of the conic, and the centre C_2 lies on it.

Further, K_2 lies on PQ_2 and K_2R_2 is parallel to K_1R_1. Also OK_1K_2, OQ_1Q_2, OR_1R_2 are straight lines.

Again Q_2R_2, Q_1R_1, being corresponding lines, meet at a point S of x, and, since R_1K_1 is parallel to SP, the triangles SPQ_1, $R_1K_1Q_1$ are similar and $SP : R_1K_1 = PQ_1 : K_1Q_1 = 2 : 1$, so that $SP = 2.R_1K_1$.

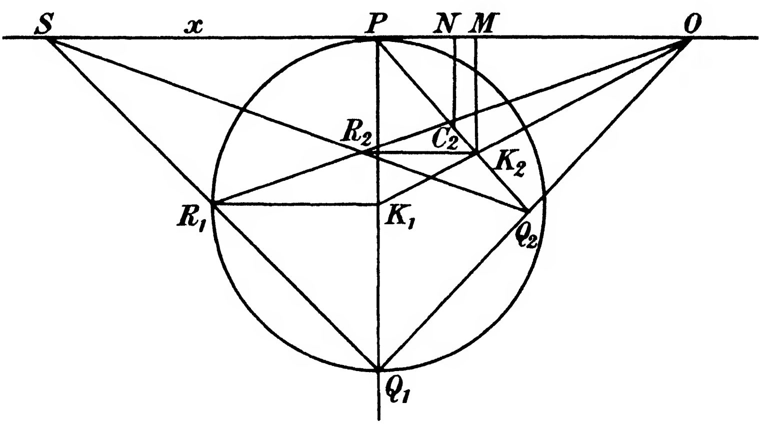


FIG. 36.

Let C_2N, K_2M be the perpendiculars from C_2, K_2 on x. We now have, by similar triangles $R_2K_2Q_2$, SPQ_2

$$R_2K_2 : Q_2K_2 = SP : Q_2P = R_1K_1 : C_2P \quad \ldots\ldots\ldots\ldots(1)$$

since $SP = 2.R_1K_1$ and C_2 is the middle point of the diameter Q_2P of the conic.

Further, the triangles R_1K_1P, R_2K_2M are clearly in perspective from O, and since two pairs of corresponding sides are parallel, the third pair by Art. 11 are also parallel and the triangles are similar. Hence, since $R_1K_1 = K_1P$, we have also

$$R_2K_2 = K_2M \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Now, from similar triangles C_2NP, K_2MP

$$K_2M : K_2P = C_2N : C_2P,$$

that is, in virtue of (2)

$$R_2K_2 : K_2P = C_2N : C_2P \quad \text{..................(3)}$$

Multiplying together (1) and (3)

$$\frac{R_2K_2^2}{Q_2K_2.K_2P} = \frac{R_1K_1.C_2N}{C_2P^2} \quad \text{..................(4)}$$

If the conic is a central conic, then, by Art. 76, since R_2K_2 is the oblique ordinate conjugate to the diameter Q_2P,

$$\frac{R_2K_2^2}{Q_2K_2.K_2P} = \pm \frac{d^2}{C_2P^2},$$

the positive or negative sign being taken according as the conic is an ellipse or hyperbola, d being the absolute length of the semi-diameter conjugate to C_2P.

Substituting into (4), we have

$$R_1K_1.C_2N = \pm d^2$$

or $$R_1K_1 = \textbf{radius of curvature} = \pm d^2/p,$$

where p is the perpendicular C_2N from the centre of the conic on the tangent at P. The interpretation of the minus sign in the case of the hyperbola is that C_2N is drawn in the sense opposite to that shown in Fig. 36, and p is to be reckoned negative. The centres of the circle and conic are then on opposite sides of the tangent.

If P is at a vertex of the conic, the circle of curvature has four-line contact with the conic; O, M, N then coincide with P, and K_2, Q_2 lie on PQ_1. The proof of equation (1) still holds good; further, we obtain at once, from the similar triangles R_2K_2P, R_1K_1P, that $R_2K_2 : K_2P = R_1K_1 : K_1P = 1 = C_2N : C_2P$, which is identical with (3), and leads to the same formula.

The result (4) can also be used to construct the circle of curvature in the case of the parabola. For we may write (4) in the form

$$\frac{R_2K_2^2}{K_2P} = Q_2K_2.\frac{Q_1P}{Q_2P}.\frac{C_2N}{C_2P},$$

since $Q_1P = 2.R_1K_1$, $Q_2P = 2.C_2P$.

If now PQ_2 meet the circle again at V (not shown in Fig. 36), then, since the angle PVQ_1 is a right angle, the triangles PVQ_1, C_2NP are similar, and $C_2N : C_2P = PV : Q_1P$, so that

$$\frac{R_2K_2^2}{K_2P} = \frac{Q_2K_2}{Q_2P}.PV \quad \text{........................(6)}$$

(6) holds good for any conic; if, however, the conic is a parabola, Q_2 is at infinity, and we may write $\dfrac{Q_2K_2}{Q_2P}=1$. Thus $PV=\dfrac{R_2K_2^2}{K_2P}=$ parameter (Art. 76) of the chords parallel to the tangent at P.

EXAMPLES VA

1. Two hyperbolas, s_1 and s_2, touch one another at A and have parallel asymptotes. Show that the line joining their centre passes through A.

If any line through A meets s_1 again at P_1 and s_2 again at P_2, prove that the tangents at P_1, P_2 are parallel.

2. The tangents at points P, Q of a circle meet at R, and A is any other point of the circle. Prove that AR and the tangent at A are harmonically conjugate with respect to AP, AQ.

Deduce that, if the tangents at points P, Q of a hyperbola meet one of the asymptotes at L, M, then PQ meets that asymptote at the middle point of LM.

3. P, Q, R are three points on a parabola. A line through P parallel to the axis meets QR at L, and a line through R parallel to the axis meets PQ at M. Show that LM is parallel to the tangent at Q.

4. If l is any given line in the plane of a parabola, and a variable diameter meets l at L and the curve at Q, and if R is the mid-point of LQ, prove that the locus of R is a parabola.

5. Show that the triangle formed by three tangents to a conic is in plane perspective with the triangle formed by their three points of contact.

6. The tangents to a hyperbola at P and Q meet an asymptote at R, S respectively. If $(RP, SQ)=U$, $(SP, RQ)=V$, show that UV is parallel to this asymptote.

7. If a straight line meet a hyperbola at P and the asymptotes at Q, R, prove that $PQ.PR=$ square of parallel semi-diameter.

8. Show that any chord of a rectangular hyperbola subtends equal or supplementary angles at the extremities of any diameter.

9. Obtain the following construction for a parabola by tangents, given four tangents a, b, c, d. On (ab, cd) take any point B. Draw through B a parallel to a meeting c at P, and a parallel to d meeting b at Q. PQ is a tangent to the parabola.

10. Show how to construct a parabola by tangents given three tangents and the direction of the axis. Construct also the vertex and axis.

11. Prove that all circles touching a conic at the same point O have their common chords with the conic, not passing through O, parallel to a fixed direction.

12. If the tangent at U meet a pair of conjugate diameters at P, P', show that $PU.UP'=CD^2$ where CD is the semi-diameter parallel to the tangent at U.

13. If PP', DD' be conjugate diameters of a hyperbola in absolute length and position and Q any point on the curve, show that $QP^2+QP'^2$ differs from $QD^2+QD'^2$ by a constant quantity.

14. The extremities A, A' of a diameter of a rectangular hyperbola are joined to a point P on the curve. Show that AP, $A'P$ describe two oppositely

equal pencils. Show also that this is not true unless A, A' are extremities of a diameter.

15. A, A' are fixed points in a plane, and a point P in the plane moves so that the bisectors of the angle APA' are parallel to fixed directions. Find the locus of P and show how to construct its asymptotes.

16. Find the locus of the vertices of the conjugate parallelograms of an ellipse.

17. From a point P on a hyperbola PN is drawn perpendicular to the transverse axis and from N a line is drawn to touch the auxiliary circle at T. Prove that $TN : PN =$ ratio of semi-transverse to semi-conjugate axis.

18. AA' is a diameter of a conic, T is a point on the tangent at A, P is the point of contact of the second tangent from T, PN is the chord through P conjugate to AA' meeting AA' at N, and TA' meets PN at Q. Prove that $PQ = QN$.

19. Show that the six points, in which the three escribed circles of a triangle touch the sides of the triangle when produced, lie on a conic.

20. What are the characteristic properties of a geometrical figure which is unaltered by orthogonal projection?

ACA', BCB' are a pair of conjugate diameters of an ellipse and P is a point on the curve; AP, $A'P$ meet BB' at Q and Q'. Show that a similar and similarly situated ellipse can be drawn through the points Q, P, Q'; and that BP, $B'P$ pass through the extremities of its diameter parallel to AA'.

21. Prove that two coplanar conics which touch at a point T correspond in a plane perspective having for axis the common tangent t at T; and show that the two conics have three-line contact at T if, and only if, the pole of perspective lies on t.

Show how to find, by a geometrical construction, the direction of the axis of the parabola which has three-line contact with a circle c at a given point T, and also touches a given line u.

22. Show that the central chord of curvature of a conic at $P = 2.CD^2/CP$, CD being the semi-diameter conjugate to CP.

23. Show that any point of a rectangular hyperbola is a point of trisection of the intercept of the normal at the point between the centre of curvature and the point where the normal meets the curve again.

24. Prove that through any point P of a conic, three circles of curvature of the conic pass other than the circle of curvature at P.

[Let PQ, PQ' be chords equally inclined to axes, CR the diameter conjugate to PQ' meeting PQ at S. $P[Q] \barwedge P[Q']$ (oppositely equal); $P[Q'] \barwedge C[R]$ (conjugate); $\therefore P[Q] \barwedge C[R]$. The three points other than P where the conic locus of S meets the original conic have their circles of curvature passing through P, for at such points Q, R, S coincide and the tangent at Q is equally inclined to the axes with PQ.]

EXAMPLES Vb

[The axes of co-ordinates are rectangular, except where otherwise stated.]

1. Given that the angle between the axes of co-ordinates is 75°, draw the hyperbola having the axes for asymptotes and passing through the point (2, 0·5).

2. Draw the conic through the points (1, 1·5), (4, 0·5), (5·8, 4·3), (6, 2), (3, 4·4) by the Pascal line method.

3. Find graphically four additional points on the conic which passes through the five points (0, 0), (0, 4), (2, 2), (1·41, 3·41), (– 1·41, 0·59), and then make a rough sketch of the curve.

4. A conic passes through the points whose co-ordinates are (2, 0), ($\frac{5}{2}$, $\frac{7}{2}$), (4, 2), and touches the line $4y = 7(x + 1)$ at the point ($\frac{5}{7}$, 3). Draw the conic.

5. Draw the parabola which touches four sides of a regular pentagon inscribed in a circle of 2″ radius.

6. A hyperbola passes through the points (1, 0), (1, 3), (6, 0), (6, 4), and has one asymptote parallel to $y = 2x$. Construct both asymptotes in position.

7. A hyperbola passes through the points (1, 0), (1, 1), (0, – 1), and has the line $x + y = 0$ for an asymptote. Draw the tangents at the three given points, and find one other point on each branch of the curve.

8. A hyperbola has the axis of y for one asymptote and touches the lines $x + y = 4$, $\frac{x}{5} + \frac{y}{2} = 1$, $\frac{x}{3} + \frac{y}{8} = -1$. Construct it by tangents.

9. ABC is an equilateral triangle of side 4 inches, and D is the middle point of BC. A hyperbola passes through A, B and D, and has its asymptotes parallel and perpendicular to AC. Construct geometrically (i) the centre, (ii) the asymptotes, (iii) the second point in which the hyperbola meets the line through B parallel to AD.

10. A parabola touches the line $y = x$ at the origin and its axis is parallel to Ox. If it passes through the point (4, – 3), draw the curve.

11. Construct the vertex and axis of the parabola through the three points (2, – 1), (3, 2), (6, 4) whose axis is parallel to the line $14x + 3y = 21$.

12. A parabola touches the axes of co-ordinates and also the lines $2y - x = 1$, $x - y = 2$. Construct its axis, the tangent at its vertex, and the points of contact of the four given tangents.

13. A parabola touches the line $y = 2x + 2$ at the point (1, 4) and also touches the axes of co-ordinates. Construct (i) the points of contact of the co-ordinate axes, (ii) the tangent at the vertex, (iii) the axis of the parabola.

14. A parabola touches the line $x = y$ at the origin, has its axis parallel to the axis of x, and passes through the point $P(-1, -4)$. Construct (i) the tangent at P, (ii) the second point where the axis of y meets the curve, (iii) the axis of the parabola.

15. Draw the parabola which touches three sides AB, BC, CD of a regular pentagon $ABCDE$ of side 1″ and whose axis is parallel to the line joining C to the middle point of AB. Construct its axis and vertex.

16. ABC is an equilateral triangle of 4″ side. P is a point on AB between A and B, distant 1″ from A. Q is a point on AC between A and C, distant 2·5″ from A.

Construct by tangents the conic which touches AB at P, AC at Q and also touches BC.

17. A parallelogram $ABCD$ has sides $AB = DC = 1$ inch, $BC = AD = 3$ inches, and the angle BAD is 60° ; E is the point on BC such that the angle ADE is 75°. An ellipse touches AB, DC at A, D respectively, and passes through E. Construct the tangent at E to this ellipse, and the second point in which the ellipse meets BC.

18. $ALA'M$ is a simple convex quadrilateral such that $AA' = 4″$, $AL = 2″$, $AM = 1\frac{1}{2}″$, $A'L = A'M = 3″$. An ellipse has A and A' for the extremities of

its major axis and passes through L. Construct (i) the tangent at L, (ii) the second point P in which the ellipse is met by LM, (iii) the fourth point in which the ellipse is met by the circle $A'LP$.

19. Draw two straight lines OX, OY inclined at 60°; on OX mark $OA=AB=2''$, and on OY mark $OC=CD=DE=1''$. A conic is drawn to touch OX, OY, AE, BD; and C is the point of contact of OY. Find the points of contact of OX, AE and BD.

20. A hyperbola has the points $(\pm 2, 0)$ for the extremities of its transverse axis and passes through the point $(5, 7)$. Construct its asymptotes geometrically and find the absolute length of its conjugate axis.

CHAPTER VI

FORMS OF THE SECOND ORDER AND SELF-CORRESPONDING ELEMENTS

85. Projective ranges and pencils of the second order. The points of a conic, like the points of a straight line, may be spoken of as forming a range, but such a range is said to be of the second order, the linear range being considered of the first order.

Similarly the tangents to a conic are said to form a pencil of the second order.

These are termed **forms of the second order,** and the conic to which they belong is called their **base.**

Ranges and pencils of the second order will be denoted by writing 2 as an index outside the bracket enclosing the typical element: thus $[P]^2$, $[p]^2$.

Now a range of the second order $[P_1]^2$ determines, at an arbitrary point A of its conic base, a flat pencil $A[P_1]$. If we vary the position of A on the conic, all the pencils $A[P_1]$ are projective with one another by Art. 37.

Similarly another range of the second order $[P_2]^2$ determines, at an arbitrary point B of its own base, a flat pencil $B[P_2]$, and the various pencils $B[P_2]$, obtained by varying B, are projective with one another.

If now any one pencil $A[P_1]$ is projective with any one pencil $B[P_2]$, then this is true of all such pencils, independently of the choice of A and B on the conics, provided the points P_1, P_2 remain unaltered. The condition in question is therefore one which involves only the relation between the ranges $[P_1]^2$, $[P_2]^2$ themselves, and, when it is satisfied, these ranges are said to be projective. It will be shown in Art. 166 that they can actually be projected into one another.

Thus two ranges of the second order are projective if the pencils which they determine at any points of their respective bases are projective.

Similarly two pencils of the second order are said to be projective if the ranges which they determine on any tangents to their respective bases are projective.

From the known properties of projective pencils and ranges of the first order given in Chapter II it follows that two corresponding triads entirely determine the relation between two projective forms of the second order. Also if we define the cross-ratio of four points on a conic as the cross-ratio of the pencil which they determine at any point of the conic and the cross-ratio of four tangents to a conic as the cross-ratio of the range which they determine on any tangent to the conic, then projective forms of the second order are equi-anharmonic and conversely.

Again, as in Chapter II, two cobasal forms of the second order cannot have more than two self-corresponding elements without being entirely coincident.

Since four elements of a range or pencil of the second order have a cross-ratio, they may form a harmonic set, when this cross-ratio is equal to -1. The condition that two such elements shall be harmonically conjugate with respect to another two is easily obtained.

Thus, let A, B, C, D be four points on a conic; let AT be the tangent at A meeting BD at T, and join AB, AC, AD. If O be any point on the conic, then (A, C) are harmonically conjugate with respect to (B, D) if $O\{ABCD\} = -1$. If we make O coincide with A, the above condition becomes $A\{TBCD\} = -1$. If AC meet BD at E, then, cutting the pencil $A(TBCD)$ by BD, we have $\{TBED\} = -1$, or E is a point on the polar of T. But the point of contact A of a tangent from T is also on the polar of T. Thus the polar of T is AE, that is, AC. Hence the pole of AC lies on BD, and **the joins of harmonic conjugates are conjugate lines for the conic.**

Conversely, if AC is conjugate to BD, let the tangent at A meet BD at T. The pole of AC lies on the tangent at A, and also (by the property of conjugate lines) on BD. Hence it must be T, and if AC meet BD at E, (T, E) are harmonically conjugate with respect to (B, D), and $A\{TBED\} = -1$, that is, $A\{TBCD\} = -1$. But this last is the cross-ratio of the four points A, B, C, D on the conic, so that (A, C) are harmonically conjugate to (B, D) on the conic.

In a similar manner, using the principle of duality, we can show that the necessary and sufficient condition for two tangents (a, c) to be harmonically conjugate to two other tangents (b, d) in the pencil of the second order formed by the tangents to a conic, is that ac and bd are conjugate points with respect to the conic. **Thus pairs of tangents from conjugate points are harmonically conjugate with respect to one another.**

86. Cross-axis and cross-centre of cobasal projective forms of second order. Let P, P' (Fig. 37) be two corresponding points of two projective ranges $[P]^2$, $[P']^2$ lying on the same conic s.

Let A, A' be any given corresponding points of these ranges. Project the range $[P']^2$ from A as vertex, $[P]^2$ from A' as vertex. The pencils $A[P']$ and $A'[P]$ are projective and they have a self-corresponding ray $A'A$. Hence they are perspective and rays AP', $A'P$ meet at U on a fixed axis x.

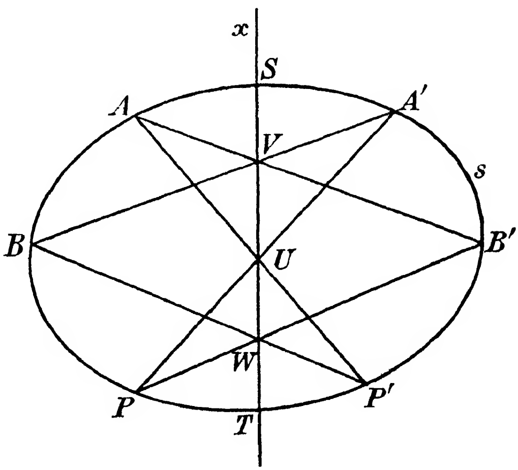


FIG. 37

This axis x is independent of the choice of the points A, A'. For let B, B' be any other pair of corresponding points. Then by the previous result AB', $A'B$ meet at V on x. Now consider the Pascal hexagon $AB'PA'BP'$. We have $(A'B, AB')$ $(AP', A'P)$ $(PB', P'B)$ are collinear. x is therefore the Pascal line and PB', $P'B$ meet at W on x. The same line x is therefore reached if we start from A and A', or if we start from B and B'.

There is thus a fixed line x on which meet the cross-joins AB', $A'B$ of any two corresponding pairs. This we shall call, as in the case of linear ranges, the **cross-axis.**

By reciprocation, or by proceeding in a manner similar to the above and using Brianchon's Theorem, we reach the result that two projective pencils of tangents to the same conic have a **cross-centre,** through which pass the joins of cross-meets $(ab', a'b)$ of two corresponding pairs.

Examples

1. Directly equal ranges on a circle may be defined as ranges in which two directly equal pencils whose vertices are on the circle meet the circle. Show that the cross axis of two such ranges is at infinity.

2. Oppositely equal ranges on a circle may be defined as ranges in which two oppositely equal pencils whose vertices are on the circle meet the circle. Show that the cross axis of two such ranges passes through the centre.

87. Self-corresponding elements of cobasal projective forms of second order. As mentioned already in Art. 85, two cobasal projective forms of the second order cannot have more than two self-corresponding elements; for if they have three, say A, B, C and if P, P' be any other pair of corresponding elements, $\{ABCP\}=\{ABCP'\}$ and as in Art. 25 $P'=P$.

These self-corresponding elements may be constructed as follows. If the cross-axis of two projective ranges of the second order $[P]^2$, $[P']^2$ lying on the same conic s meet s at points S, T (Fig. 37) the points S, T are self-corresponding points of the ranges $[P]^2$, $[P']^2$.

For by the property of the cross-axis AT, $A'T$ meet s again at a pair of corresponding points. But they both meet s again at T. Hence a pair of corresponding points coincide at T, or T is self-corresponding. Similarly S is self-corresponding.

If the cross-axis x is itself a tangent to s, the self-corresponding points S, T coincide. If x do not meet the conic at real points, there are no real self-corresponding points.

Reciprocating, we have the theorem: the self-corresponding lines of two projective pencils of the second order belonging to the same conic are the tangents from the cross-centre. There are two real self-corresponding lines if the cross-centre is outside the conic: these coincide if the cross-centre is on the conic. If the cross-centre be inside the conic there are no real self-corresponding lines.

88. Two corresponding elements of two cobasal projective forms determine with the self-corresponding elements a constant cross-ratio. It will be sufficient to prove this for two projective ranges on the same conic, since all other cases can clearly be made to depend upon this. Now from Fig. 37, if AP', $A'P$ be two chords meeting on ST, then A', P' are corresponding points of the ranges determined by the triads S, T, A; S, T, P. Hence, on this conic, the cross-ratio of the four points $STAA'$ is equal to the cross-ratio of the four points $STPP'$, which proves the theorem.

89. Construction of self-corresponding points. The results of Art. 87 provide us with a construction for determining the self-corresponding elements of two cobasal projective forms of the first order.

Thus let there be two projective pencils having a common vertex O: let $a_1b_1c_1$; $a_2b_2c_2$ be two corresponding triads.

Describe any conic (in practice a circle will be a convenient conic to use) passing through O, and meeting a_1, b_1, c_1, a_2, b_2, c_2 at A_1, B_1, C_1, A_2, B_2, C_2 respectively. Construct the cross-axis of the ranges of the second order on this conic defined by $A_1B_1C_1$, $A_2B_2C_2$. This cross-axis is obtained from any two pairs of cross-joins (A_1B_2, A_2B_1) and (A_1C_2, A_2C_1).

The points S, T where this cross-axis meets the conic are self-corresponding points of the ranges of second order. The rays OS, OT are then self-corresponding rays of the given pencils of first order, since corresponding rays of these pencils pass through corresponding points of the ranges of second order.

On the other hand let there be two projective ranges on the same straight line u, defined by corresponding triads $A_1B_1C_1$, $A_2B_2C_2$.

Describe any conic (here again in practice a circle) touching u. From A_1, B_1, C_1, A_2, B_2, C_2 draw tangents a_1, b_1, c_1, a_2, b_2, c_2 to this conic. Construct the meet of the joins (a_1b_2, a_2b_1) and (a_1c_2, a_2c_1). This is the cross-centre. The two tangents from the cross-centre meet u at the self-corresponding points of the given ranges.

Otherwise thus: the two ranges may be projected from any vertex and the self-corresponding rays of the concentric projective pencils so formed may be found by the construction given at the beginning of this article. They meet u at the self-corresponding points of the ranges.

90. Intersections of a straight line with a conic given by five points. Let O, O', A, B, C be the five points on the conic, u any straight line.

The conic is the product of the two projective pencils defined by $O(ABC)$, $O'(ABC)$.

If OA, OB, OC meet u at A_1, B_1, C_1 and $O'A$, $O'B$, $O'C$ meet u at $A_2B_2C_2$, the pencils $O(ABC)$, $O'(ABC)$ determine upon u two projective ranges of which $A_1B_1C_1$, $A_2B_2C_2$ are corresponding triads.

Find the self-corresponding points of these ranges on u by either of the methods given in the last article. Let these be S, T. Then OS, $O'S$ are corresponding rays of the pencils $O(ABC)$, $O'(ABC)$.

Therefore S is a point on the conic.

Similarly T is a point on the conic.

Hence S, T are the intersections of u with the conic.

91. Directions of asymptotes of a conic given by five points. If in the construction of the preceding Article the line u be the line at infinity $A_1B_1C_1$, $A_2B_2C_2$ are at infinity. Let now P be any point of the conic, and let OP, $O'P$ meet u^∞ at $P_1{}^\infty$, $P_2{}^\infty$ respectively. Since $O[P]\barwedge O'[P]$, therefore $[P_1{}^\infty]\barwedge[P_2{}^\infty]$. The points S^∞, T^∞, in which the conic meets u^∞, are then the self-corresponding points of the projective ranges $[P_1{}^\infty]$, $[P_2{}^\infty]$. Hence OS^∞, OT^∞ are the self-corresponding rays of the projective pencils $O[P_1{}^\infty]$, $O[P_2{}^\infty]$, that is, $O[P]$, $[p']$, where p' is the line through O parallel to OP. Now OA, OB, OC of the pencil $O[P]$ correspond respectively to $OA_2{}^\infty$, $OB_2{}^\infty$, $OC_2{}^\infty$ of the pencil $[p']$. Find by the method of Art. 89 the self-corresponding rays of the projective pencils through O defined by these triads; these self-corresponding rays pass through S^∞, T^∞, and therefore give the directions of the asymptotes. The asymptotes are then constructed in position by the method of Art. 68.

92. Construction of the parabolas through four given points. Let O, O', A, B be the four given points (Fig. 38). Through O draw any circle meeting OA, OB at A_1, B_1 and the parallels through O to $O'A$, $O'B$ at A_2, B_2.

Then if P is any point on the parabola and P_1, P_2 are the points where OP and the parallel through O to $O'P$ meet the circle, $[P_1]^2$, $[P_2]^2$ are two projective ranges on the circle whose self-corresponding points are the points corresponding to the points at infinity on the curve, since when P is at infinity OP, $O'P$ are parallel.

In the case of the parabola the points at infinity are coincident because the line at infinity touches the curve. Hence the self-corresponding points of the ranges $[P_1]^2$, $[P_2]^2$ are coincident and the cross-axis touches the circle (Art. 87).

But we know one point on the cross-axis, namely the point U where A_1B_2 meets A_2B_1. The cross-axis is therefore either of the two tangents from U to the circle. The join of O and the point of contact of the cross-axis with the circle gives the direction of the point at infinity on the parabola, or the direction of the axis. Having the direction of the axis and four points on the curve we may construct the parabola by the method of Art. 71, or more directly as follows. Take any point Q on the cross-axis. Join QB_2 meeting the circle at P_1, QB_1 meeting the circle at P_2. The parallel through O' to OP_2 meets OP_1 at a point P on the parabola.

Since two tangents can be drawn to a circle from U, the problem is in general capable of two solutions. These solutions are coincident if U be on the circle. In this case either A_1 and B_1 (or A_2 and

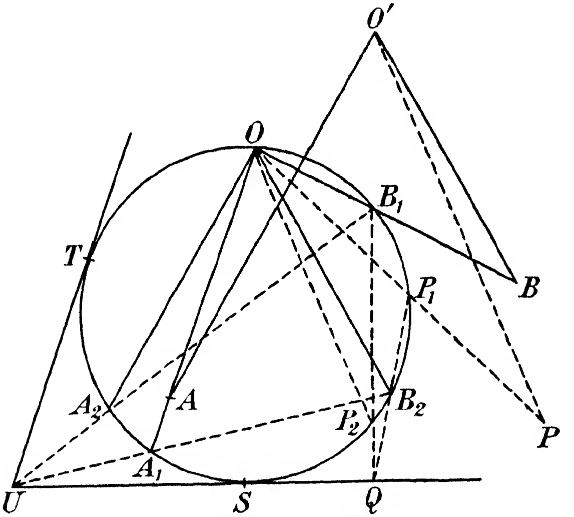


FIG. 38.

B_2) coincide, that is, three of the given points are collinear and the conic then degenerates into two parallel straight lines, which is a special case of a parabola ; or else A_1 and A_2 (or B_1 and B_2) coincide ; A (or B) is then at infinity, so that three points and the direction of the axis are given and the parabola can be drawn by Pascal's Theorem. If U be within the circle there are no real solutions to the problem.

93. Rectangular hyperbola through four points. Case of failure. The same principle will enable us to construct the rectangular hyperbola through four given points O, O', A, B. Draw a circle through O and find the points A_1, B_1, A_2, B_2 and the point U on the cross-axis by the same construction as before. Now since in the rectangular hyperbola the asymptotes are to be at right angles the self-corresponding rays of the pencils $O[P_1]$, $O[P_2]$ are at right angles, that is, they meet the circle at the extremities of a diameter; or the cross-axis of $[P_1]^2$, $[P_2]^2$ is a diameter. Hence join U to the centre of the circle, and we have the cross-axis required. The joins of O to its intersections with the circle give the directions of the asymptotes. Having these and four points on the curve we can construct the curve by Art. 68 or directly from the present construction as explained in the last article.

If U be at the centre of the circle, any diameter may be taken as the cross-axis and an infinite number of rectangular hyperbolas may be drawn through the four points. In this case A_2B_1, A_1B_2 being diameters, OA, OB are perpendicular to OB_2, OA_2, that is, to $O'B$, $O'A$, or O is the orthocentre of the triangle $O'AB$. It is easy to prove that when this is so any one of the four given points is the orthocentre of the triangle formed by the other three.

94. Tangents from any point to a conic given by five tangents. Let t, t', a, b, c be five tangents to a conic.

Let A, B, C be the points where t meets a, b, c,

A', B', C' ,, ,, ,, t' ,, a, b, c.

Let O be any point in the plane.

If p be any tangent to the conic meeting t at P and t' at P', the ranges $[P]$, $[P']$ are projective: hence the pencils $O[P]$, $O[P']$ are projective.

If p passes through O, OP and OP' are coincident.

Therefore the tangents to the conic through O are the self-corresponding rays of the pencils $O[P]$, $O[P']$.

Determine these self-corresponding rays from the triads $O(ABC)$, $O(A'B'C')$ by the method of Art. 89; then these give the tangents required.

EXAMPLES VIA

1. A conic passes through five points O, O', A, B, C. Show how to construct graphically its intersections with any circle through O, O' without drawing the conic. Prove that the common chord not passing through OO' is always real, even when the circle does not meet the conic again in real points.

2. A conic is given by five points. Without drawing the curve find a test to determine whether it is an ellipse, hyperbola or parabola.

3. Prove that two conics can be drawn through four given points such that their asymptotes make an angle a with one another: and show how to construct them.

[In the construction of Art. 92 the cross-axis must cut off a constant arc from the circle through O and therefore touches a circle concentric with this circle.]

4. Investigate the nature of the simple quadrilateral formed by four points if it is impossible to draw a real parabola through them.

5. The lines joining a variable point P of a given conic k to two fixed points A and B meet k again at Q, R respectively. Prove that if Q_1R_1, Q_2R_2 be any two positions of the line QR, then Q_1R_2, Q_2R_1 meet at a point of the line AB, and show that $[Q]^2 \barwedge [R]^2$.

6. In Ex. 5 prove that QR passes always through a third fixed point C if, and only if, the triangle ABC is self-polar for k.

7. Given five points on a conic draw the tangents to it from any point in the plane.

[Find where two rays through the point cut the conic. Hence construct the polar.]

8. Given five tangents to a conic, find its intersections with any given straight line in its plane.

9. A_1B_1, A_2B_2 are two corresponding pairs of points of two collinear projective ranges. Given that the self-corresponding points of the two ranges are coincident, find the possible positions of the point at which they coincide.

10. Prove that the projective relation which transforms three real points A, B, C of a line l into B, C, A respectively, associates the points of l in triads P, Q, R permuted cyclically by the projective relation; and that the projective ranges so defined have no real self-corresponding point.

If U, V are fixed points coplanar with l and the projective pencils defined by $U(ABC)$ and $V(BCA)$ have the conic k for their product, prove that rays $U(PQR)$ [or $V(PQR)$] meet k again at points $P'Q'R'$ such that $P'Q'R'$ correspond to $Q'R'P'$ respectively in two projective ranges on k, having l for their cross-axis.

11. Two conics have three-point contact at O. A ray through O meets the conics again at P_1, P_2 and the tangents at P_1, P_2 meet the tangent at O at Q_1, Q_2. Show that the ranges $[Q_1]$, $[Q_2]$ are projective, and have no self-corresponding point other than O.

EXAMPLES VIb

[The axes of co-ordinates are rectangular, except where otherwise stated.]

1. A, B, C, D are four points on a straight line at unit distance apart in order. ABC, DCA define two projective ranges. Construct the self-corresponding points of these ranges.

2. Draw an indefinitely long line Ox, and on it take A, B, C such that $AB=3$, $BC=2$. Take also on Ox three points A', B', C' such that $CC'=6$, $CB'=10$, $CA'=12$. It is required to find the position of a point F on Ox such that the cross-ratios $\{ABCF\}$ and $\{A'B'C'F\}$ shall be the same. Verify your construction by algebraic calculation.

3. O, O' are two points 4″ apart: through O are drawn three rays OA, OB, OC making with OO' angles of 90°, 60°, 30° (counter-clockwise); and through O' are drawn three rays $O'A$, $O'B$, $O'C$ making with $O'O$ angles of 30°, 15°, 75° (clockwise).

Without drawing the curve construct the asymptotes of the locus of intersections of corresponding rays of the projective pencils defined by the triads $O(ABC)$, $O'(ABC)$.

4. Find the directions of the axes of the parabolas which can be drawn through the four points whose co-ordinates are

$$(-0{\cdot}5,\ -1{\cdot}5),\ (4,\ 0),\ (-0{\cdot}9,\ -0{\cdot}4),\ (7{\cdot}5,\ -1{\cdot}5).$$

5. Construct the rectangular hyperbola through the four points (0, 0), (0, 2), (1, 0), (1, 3).

6. The angle between the axes of x, y being 45° a conic touches the lines $2x+y=2$, $3x+10y=30$, $x+y=5$ and the axes. Without drawing the curve, construct the two tangents to it from the point (4, -3).

7. The following points are given: $O(0, 0)$, $O'(3, 0)$, $A(-1, 4)$, $B(2, 2)$, $C(6, 5)$. $O(ABC)$, $O'(ABC)$ define two projective pencils. Construct the rays of the first pencil which are parallel to the corresponding rays of the second pencil.

8. M, N are the middle points of the sides AB, AD of a square $ABCD$, of side 2 inches. Construct the two points in which the diagonal BD is met by the conic which touches AB, AD at M, N respectively, and passes through C.

9. A hyperbola passes through the points (0, 0), (4, 0), (4, 8), (3, 3), (-1, -5). Construct its asymptotes, the tangent at (0, 0), and the vertices.

CHAPTER VII

INVOLUTION

95. Involution. Let P, P' be two distinct corresponding elements (denoted by italic capitals, but here not restricted to mean points) of two cobasal projective forms ϕ, ϕ'.

Then in general if P be considered as an element of ϕ' the element of ϕ which then corresponds to P is not P', but some other point.

It may, however, happen that P' corresponds to P, whether P be considered as belonging to ϕ or as belonging to ϕ'. P and P' are then said to correspond doubly.

In this case every other pair of corresponding elements Q, Q' also correspond doubly. For since by Art. 21 a cross-ratio is not altered if we interchange two of its elements, provided the other two be also interchanged,

$$\{PP'QQ'\} = \{P'PQ'Q\}.$$

But by hypothesis $PP'Q$, $P'PQ'$ are corresponding triads of ϕ, ϕ' respectively. Hence the above equation expresses the fact that to Q' of ϕ corresponds Q of ϕ^1 or Q, Q' correspond doubly.

Two cobasal projective forms, in which every element corresponds doubly, are said to be in **involution,** or to form an involution on their base. The corresponding elements are spoken of as **mates** in the involution.

Examples

1. Show that any line through the cross-centre of two projective pencils meets the two pencils in an involution; find the mate of the cross-centre in this involution.

2. Show that if O is any point on the cross-axis of two projective ranges $[P_1]$, $[P_2]$, the pencils $O[P_1]$, $O[P_2]$ form an involution, and find the mate of the cross-axis in this involution.

3. If (P, P') are mates in an involution range on a straight line, show that, if O is a point outside the line, OP, OP' are mates in an involution pencil.

4. If mates in an involution pencil vertex O meet a conic through O in points P, P' and p, p' are the tangents to the conic at P, P', show that (P, P') are mates in an involution range on the conic, and (p, p') are mates in an involution pencil of the second order.

5. Prove that an involution projects into an involution.

6. From the result of Ex. 1 above prove that the three pairs of opposite sides of a complete quadrangle meet any straight line in three pairs of mates of an involution.

7. From the result of Ex. 2 above prove that the lines joining a given point to the three pairs of opposite vertices of a complete quadrilateral form three pairs of mates of an involution pencil.

8. If two rays through a fixed point O are equally inclined to a fixed direction, show that they are mates in an involution pencil.

9. If (A, A'), (B, B'), (C, C') be three pairs of mates of an involution, and A, A' are harmonically conjugate with regard to B and C, prove that they are also harmonically conjugate with regard to B' and C'.

96. Two pairs of mates determine an involution. Let (P, P'), (Q, Q') be the two pairs of mates. Then the triads $PP'Q$, $P'PQ'$ define two projective forms which are in involution since one pair of elements, namely P, P', correspond doubly. The involution is therefore determined.

Note that *one* pair of mates is insufficient; for *two* pairs of corresponding points (P, P'), (P', P) are not enough to determine two projective forms.

97. Double elements. Since two cobasal projective forms have two self-corresponding elements, an involution will have two self-corresponding elements, each of which is its own mate. These may or may not be real.

They are called the double elements of the involution. Since a double element is equivalent to a pair of mates, an involution is entirely given by its double elements.

There cannot be more than two double elements, since projective forms with three self-corresponding elements are identical.

An involution with real double elements is said to be **hyperbolic**; one which has no real double elements is said to be **elliptic.**

98. Any pair of mates are harmonically conjugate with regard to the double elements. For let (P, P') be a pair of mates; A, B the double elements. Then the elements $APBP'$ correspond to $AP'BP$ or

$$\{APBP'\} = \{AP'BP\}.$$

The set $APBP'$ are therefore equi-anharmonic with themselves, P and P' being interchanged: therefore (Art. 27) P and P' are harmonically conjugate with regard to A, B.

In the above the double elements have been assumed to be distinct. That this must necessarily be the case can be proved as follows.

Let A be a double element of an involution, P and P' any noncoincident pair of mates (which must exist if there is to be an involution at all). Then the triads A, P, P' and A, P', P determine the projective relation between the mates. Let now B be the point harmonically conjugate to A with respect to P, P'. Then since, by hypothesis, A, P and P' are all distinct, B must be distinct from A.

But we have, since $\{APBP'\}=\{AP'BP\}$ by Art. 27, that B corresponds to itself in the above projective relation, so that it is a double element of the involution, distinct from A.

It should be noticed that, in the case where cobasal projective forms are *not* in involution, the self-corresponding points can coincide without involving the disappearance of the general relation between corresponding elements.

99. Involution on a straight line. Centre of involution. Consider now the case of an involution on a straight line. Let O be the mate of the point O'^{∞} at infinity on the straight line. O is called the **centre of involution.** If (P, P'), (Q, Q') be two pairs of mates, we have

$$\{OPO'^{\infty}Q\}=\{O'^{\infty}P'OQ'\},$$

or
$$\frac{OP.O'^{\infty}Q}{OQ.O'^{\infty}P}=\frac{O'^{\infty}P'.OQ'}{O'^{\infty}Q'.OP'}, \quad i.e. \ \frac{OP}{OQ}=\frac{OQ'}{OP'},$$

therefore $OP.OP'=OQ.OQ'=$constant for the involution.

If Q, Q' coincide with one of the double points A, B we have

$$OP.OP'=OA^2=OB^2.$$

In a hyperbolic involution A, B are real, thus OA^2, OB^2 are positive and $OP.OP'$ is positive. Conversely, if $OP.OP'$ is positive, A, B are real. In an elliptic involution, however, $OP.OP'$ is negative and conversely.

Since $OA^2=OB^2$, O is midway between the double points.

An important particular case arises when the point at infinity is a double point. In this case O, the centre of involution, is itself at infinity. If A is the other double point, then, by Art. 98, P and P' are harmonically conjugate with respect to A, O^{∞}, so that A is the middle point of PP'.

EXAMPLES

1. Show that, if x, x' be the distances of two mates in an involution on a straight line from a fixed origin in the line, then

$$Axx'+B(x+x')+C=0,$$

A, B, C being constants.

2. Prove that conjugate points with regard to a circle on a diameter form an involution whose centre is the centre of the circle.

100. Relation between the mutual distances of six points in involution. Let (A_1, A_2), (B_1, B_2), (C_1, C_2) be three pairs of mates of an involution.

Then $$\{A_1A_2B_1C_1\}=\{A_2A_1B_2C_2\},$$

or, writing out the cross-ratios,

$$\frac{A_1A_2.B_1C_1}{A_1C_1.B_1A_2}=\frac{A_2A_1.B_2C_2}{A_2C_2.B_2A_1}.$$

Cancelling out $A_1A_2(=-A_2A_1)$ and re-arranging, we have

$$B_1C_1.C_2A_2.A_1B_2=-B_2C_2.C_1A_1.A_2B_1.$$

Now since mates in an involution have symmetrical properties, we may, in this result, interchange the suffixes 1 and 2 belonging to any letter A, B or C, and the result is still true. It may therefore be stated generally in the following form,

$$(BC.CA.AB)_{1,2}=-(BC.CA.AB)_{2,1},$$

where $(BC.CA.AB)_{1,2}$ indicates any distribution of suffixes such that a 1 and a 2 go to each letter, and $(BC.CA.AB)_{2,1}$ the same distribution with suffixes interchanged.

101. Involution flat pencil. In an involution pencil there is no special ray corresponding to the centre of an involution range, for no ray is the analogue of the point at infinity.

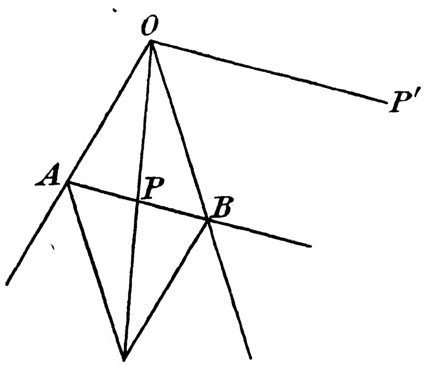


FIG. 39.

If OA, OB are the double rays, (OP, OP') a pair of mates, then cutting the pencil by a straight line parallel to OP', which meets the double rays at A and B (Fig. 39), AB is bisected at P by OP, since OP, OP' are harmonic conjugates with regard to OA, OB and therefore P and the point at infinity on OP' are harmonic conjugates with regard to A, B. Hence if the parallelogram whose sides are OA, OB be completed, its diagonals are parallel to a pair of mates.

If the double rays are at right angles, every such parallelogram is a rectangle. Its diagonals are equally inclined to the sides of the rectangle, therefore if the double rays are at right angles, they bisect the angles between any pair of mates.

102. Relation between six rays of an involution. Proceeding as in Art. 100, we have, if (OA_1, OA_2), (OB_1, OB_2), (OC_1, OC_2) are three pairs of mates of an involution pencil,

$$O\{A_1A_2B_1C_1\} = O\{A_2A_1B_2C_2\},$$

and, using the expression for the cross-ratio of a pencil in terms of the angles made by the rays (Art. 22),

$$\frac{\sin A_1OA_2.\sin B_1OC_1}{\sin A_1OC_1.\sin B_1OA_2} = \frac{\sin A_2OA_1.\sin B_2OC_2}{\sin A_2OC_2.\sin B_2OA_1},$$

whence

$$\sin B_1OC_1.\sin C_2OA_2.\sin A_1OB_2$$
$$= -\sin B_2OC_2.\sin C_1OA_1.\sin A_2OB_1,$$

and interchanging suffixes as in Art. 100 we have the general result

$$(\sin BOC.\sin COA.\sin AOB)_{1,2}$$
$$= -(\sin BOC.\sin COA.\sin AOB)_{2,1},$$

where the suffixes 1, 2 on the left-hand side indicate that a 1 and a 2 are to be assigned to each of the three letters A, B, C, the order being arbitrary.

103. Involution of points on a conic. We have already considered (Art. 85) projective ranges on a conic. Like other projective forms, these will form an involution if any one pair of corresponding elements correspond doubly.

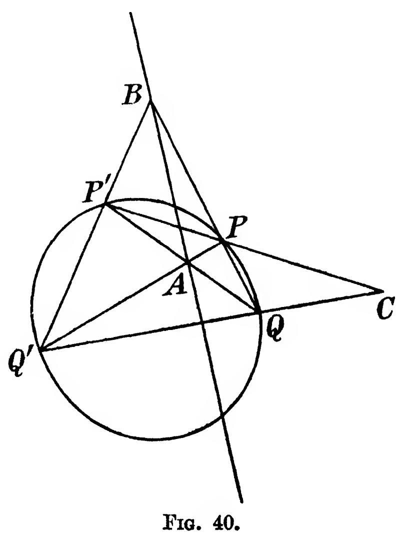


Fig. 40.

Let (P, P'), (Q, Q') (Fig. 40) be two pairs of mates in an involution of points on a conic s. Then in the projective ranges of the second order which define the involution, we have P, P', Q, Q' corresponding to P', P, Q', Q respectively.

Now two such projective ranges of the second order have a cross-axis. This we obtain from the meets of cross-joins $(PQ', P'Q) = A$; $(PQ, P'Q') = B$.

Then, by Art. 86, AB is the cross-axis and is a fixed line, independent of the choice of P, P', Q, Q'.

Let now PP', QQ' meet at C. By Art. 48, C is the pole of AB with regard to s, and is therefore a fixed point. We thus obtain the following theorem :

The joins of mates in an involution of points on a conic pass through a fixed point C, which is called the **centre** of the involution on the conic. The line AB, which is the polar of the centre of the involution, and is also the cross-axis of the projective ranges of the second order which lead to the involution, is called the **axis** of the involution.

Clearly P and P' coincide at the point of contact of a tangent from C to the conic. This point of contact lies on the polar of C, that is, on the axis of involution. Hence the double points of the involution are the points where the axis of involution meets the conic. There are clearly no real double points if C is inside the conic.

Conversely, if through a fixed point C, we draw lines CPP' to meet a conic s at P, P', then PP' are mates in an involution on the conic. For, let AA', BB' be two chords of the conic through C; the pairs of mates (A, A'), (B, B') define an involution on the conic. If (P, P') be any other pair of mates in this involution, PP' passes through the involution centre. But this is determined by the two joins AA', BB' and so must be identical with C. Hence PP' passes through C, that is, rays through C meet the conic in pairs of mates of this involution.

Examples

1. Three chords A_1A_2, B_1B_2, C_1C_2 of a circle are concurrent. If O be the centre of the circle, prove the relation

$$\sin \tfrac{1}{2}B_1OC_2 . \sin \tfrac{1}{2}C_1OA_1 . \sin \tfrac{1}{2}A_2OB_2 = -\sin \tfrac{1}{2}B_2OC_1 . \sin \tfrac{1}{2}C_2OA_2 . \sin \tfrac{1}{2}A_1OB_1$$

and similar relations.

2. If (A, B) be the double elements of an involution in which (P, P') (Q, Q') are pairs of mates, prove that (A, B) are mates in the involutions determined by the mates (P, Q'), (P', Q) or (P, Q), (P', Q').

3. A, B are two fixed points on a conic, P a variable point. Prove that the condition that $A[P]$, $B[P]$ determine an involution on any line x is that AB and x are conjugate with regard to the conic.

4. Prove that the locus of the middle points of the chords of a parabola joining mates in an involution on the parabola is another parabola which passes through the double points of the involution.

5. If O_1 be a fixed point on the cross-axis x of two projective ranges $[P_1]^2$, $[P_2]^2$ on the same conic k, and O_1P_1 meet k again at P, prove that PP_2 passes always through a fixed point O_2 of x.

Show further that, if O_1, O_2 be conjugate with respect to k, the ranges $[P_1]^2$, $[P_2]^2$ are in involution.

6. O is a point on a conic, OP, OQ are two lines equally inclined to the tangent at O, meeting the conic again at P, Q. Show that PQ passes through a fixed point on the tangent at O.

104. Involution of tangents to a conic. By reciprocating the theorems of Art. 103 we obtain the results: mates in an involution of tangents to a conic meet on a fixed line, which we call the involution axis. Also joins of cross-meets $(pq, p'q')$, $(pq', p'q)$ pass through a fixed point, which we call the involution centre. In this, as in other theorems on reciprocation, the reader will find it a useful exercise to construct the proof of the reciprocal theorem from that of the given theorem, by reciprocating each step.

The double tangents of the involution are clearly the tangents at the points where the involution axis meets the conic. Also, as in the case of the range, the centre and axis of involution are pole and polar with regard to the conic.

From two pairs of mates (pp'), (qq') the centre and axis of involution are at once constructed and either of these will give the double tangents.

EXAMPLES

1. A, B are fixed points on a fixed tangent a to a conic s. P, P' are harmonically conjugate with regard to A, B. If p, p' be the tangents from P, P' to s, show that pp' lies on a fixed straight line.

2. P is a point on a fixed straight line u, which meets a conic s at A, B. The tangents from P to s meet the tangent t to s parallel to the tangent at A or B at P_1, P_2. If C be any fixed point on t, prove that

$$CP_1 + CP_2 = \text{constant}.$$

105. Construction of double elements of an involution. The property of the centre and axis of an involution of points on a conic (Art. 103) provides a simple construction for the double elements of an involution range on a straight line, or of an involution flat pencil, when two pairs of mates are given.

Let (P, P') (Q, Q') be two pairs of mates in an involution on a straight line x. Join the pairs of mates in this involution to a fixed point O outside x; we obtain an involution flat pencil in which (OP, OP') (OQ, OQ') are pairs of mates, the double points A, B of the given involution corresponding to the double rays OA, OB of the involution pencil.

Describe any conic k, which is conveniently taken to be a circle, through O. Then the involution flat pencil with vertex O determines an involution range on k. If OP, OP', OQ, OQ' meet k again at P_1, P_1', Q_1, Q_1' respectively, then (P_1, P_1') (Q_1, Q_1') are pairs of mates in the involution on k, and the double points A_1, B_1 of this last involution are the points where OA, OB meet k again.

Construct the axis of the involution on k, by joining P_1Q_1, $P_1'Q_1'$ meeting at X, and P_1Q_1', $P_1'Q_1$ meeting at Y. Join XY, then, by Art. 103, XY is the axis required, and meets k at A_1, B_1. These being known, OA_1 and OB_1 meet x at the double points A, B of the given involution.

If, instead of being given an involution range on a straight line, we are given an involution flat pencil with vertex O, and two pairs of mates (p, p'), (q, q') of this pencil, all we have to do is to use the latter part of the previous construction, P_1, P_1', Q_1, Q_1' being the points at which p, p', q, q' meet any conic k passing through O. Then OA_1, OB_1 give the double rays of the involution flat pencil.

106. An involution is elliptic or hyperbolic according as a pair of mates are, or are not, separated by any other pair of mates. Consider first an involution on a circle k, in which (P_1, P_1') (Q_1, Q_1') (Fig. 41) are pairs of mates.

If Q_1 and Q_1' lie on opposite arcs bounded by P_1, P_1' (Fig. 41 (a)), they are separated on the circle by P_1, P_1', and conversely, P_1, P_1' are separated by Q_1, Q_1'. In this case P_1P_1', Q_1Q_1' meet at a point C inside the circle. But C is the centre of the involution, and the double points are the points of contact of tangents from C. Since C is inside the circle, no real tangents can be drawn from C to the circle, and the involution on the circle has no real double points and is elliptic.

If, on the other hand, Q_1 and Q_1' lie on the same arc bounded by P_1, P_1' (Fig. 41 (b)), P_1P_1' and Q_1Q_1' meet outside the circle. Real tangents can be drawn from C to the circle and their points of contact give real double points, so that the involution is hyperbolic.

Similarly a pair of points Q, Q' of a straight line x are said to be separated by the pair P, P' (Fig. 41 (a)), if one of Q, Q' lie in the finite segment PP' and the other outside this segment. In this case the pair P, P' are also separated by Q, Q'.

If O be any point not lying on x, the lines OQ, OQ' are separated by the lines OP, OP' if, and only if, the points Q, Q' are separated by P, P'.

Further, if a circle k through O be met again by OP, OP', OQ, OQ' at P_1, P_1', Q_1, Q_1' respectively, then, on the circle, P_1, P_1' are separated by Q_1, Q_1' if, and only if, in the pencil through O, the lines OP, OP' are separated by OQ, OQ'.

The involution determined on the original line by the pairs (P, P') and (Q, Q') is projected from O by the involution pencil in which OP, OP' and OQ, OQ' are pairs. This determines on the circle k the involution range in which (P_1, P_1') and (Q_1, Q_1') are pairs of mates. Any double point of the involution on the circle is projected from O by a double ray of the involution pencil, cutting the line x at a double point of the first involution. These three involutions are therefore elliptic or hyperbolic together. They are elliptic if, and only if, on the circle, P_1, P_1' are separated by Q_1, Q_1'; in which case OP, OP' are separated by OQ, OQ' and P, P' are separated by Q, Q'.

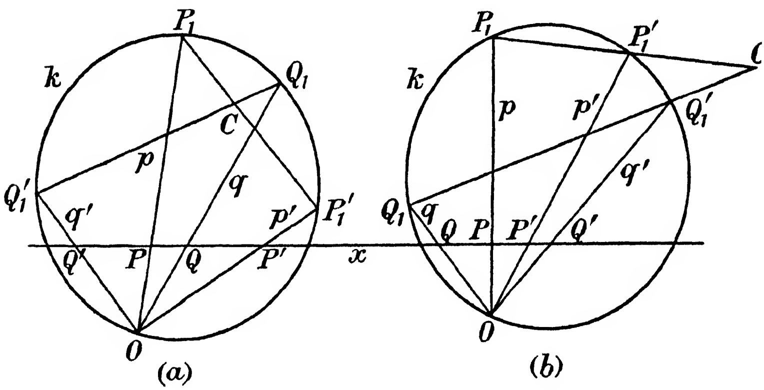


FIG. 41.

Finally consider an involution on a general conic, where (P, P') (Q, Q') are two pairs of mates. Projecting this involution by an involution pencil through any point O on the conic, the involution on the conic and the involution pencil are elliptic and hyperbolic together. The points P, P' will be said to be separated on the conic by Q, Q', if the rays OP, OP' in the pencil are separated by OQ, OQ'.

In the case where the conic is a hyperbola and P, P' are points on opposite branches, it is impossible to obtain a continuous arc joining P, P'. In this case, if I^∞, J^∞ are the points at infinity on the asymptotes, the two arcs from P to I^∞ on one branch and from I^∞ to P' on the other branch form together one arc PP'; and the two arcs from P to J^∞ on the first branch and from J^∞ to P'

on the second branch form together the complementary arc PP'. If Q, Q' are to be separated by P, P', Q must lie on one, and Q' on the other, of these two arcs PP'.

107. Common mates of two cobasal involutions. Consider two involution ranges on the same conic: the problem is to find their common mates, if any. It is clear that every other case can be reduced to this one. For two involution ranges on the same straight line can be projected from any point O by two concentric involution pencils, and two such pencils meet a conic through O in two involutions on the conic. Again, two involutions of tangents to a conic determine corresponding involutions of their points of

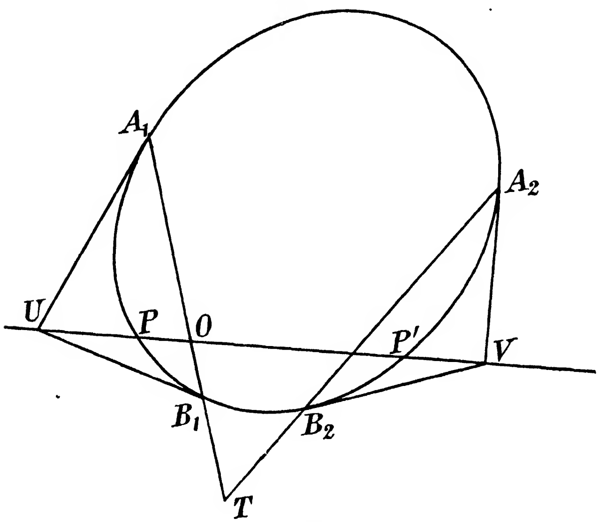


FIG. 42.

contact. If, therefore, we find the common mates of two involutions of points on a conic, these will enable us to construct the common mates in the other cases.

Now let U and V (Fig. 42) be the centres of the two given involutions of points on the conic k. Join UV, meeting the conic at P, P'. Then, since PP' passes through both U and V, P and P' are clearly mates in both involutions, and moreover, are the only points which satisfy the condition.

The problem has therefore a real solution if UV meets the conic k in real points. This always happens if one at least of U or V lies inside the conic, that is, if one of the given involutions is elliptic.

If both the given involutions are hyperbolic, and A_1, B_1 ; A_2, B_2 are their double points, then A_1B_1, A_2B_2 are the polars of U, V with respect to k. Their intersection T is then the pole of UV and the common mates are the points of contact of tangents from T. But these are the double points of the involution upon k defined by the pairs of mates (A_1, B_1), (A_2, B_2). These double points are real if the last-named involution is hyperbolic, that is, if A_1, B_1 are not separated by A_2, B_2 (Art. 106). If, however, A_1, B_1 are so separated, T lies inside k and there are no real common mates.

108. Harmonic pairs of mates. If (P, P'), (Q, Q') are two pairs of mates in an involution (of any kind) such that (Q, Q') are harmonically conjugate with respect to (P, P'), then these two pairs will be said to be **harmonic pairs,** and either is harmonic to the other.

We will now show that, in every elliptic involution, any given pair of mates has one harmonic pair and one only.

As before, it will be sufficient to prove the proposition for an involution of points on a conic k.

In Fig. 42, let O be the centre of an elliptic involution upon k; then O lies inside k. Let (A_1, B_1) be any given pair of mates in this involution, and let U be the pole of A_1B_1, then A_1, B_1 are the double points of the involution of which U is the centre. Join OU meeting k at P, P'. Then P, P' are mates in the involution of centre O; but they are also mates in the involution of centre U, and so are harmonically conjugate with respect to A_1, B_1. Hence they form a pair harmonic to (A_1, B_1).

Also there can be no other pair harmonic to (A_1, B_1), for such a pair must be mates in the involution in which A_1, B_1 are double points, and therefore must lie on a line through U; and since they must also lie on a line through O, they must lie on OU.

If the given involution were hyperbolic, so that its centre is outside k, as T in Fig. 42, then, if A_1, B_1 are real, U must also lie outside k. Now, since T lies on the polar of U, T and U are conjugate points for the conic. But, if the line joining two such conjugate points were to meet the conic at real points Q, R, then, of T and U, one must lie inside and one outside QR, so that one at least would have to be inside the conic, which is not the case. Hence TU cannot meet k in real points and the pair (A_1, B_1) has no real harmonic pair.

109. Rectangular involution. Rays at right angles through a point O determine an involution pencil through O. For, let OP, OP' be two rays at right angles. Clearly, if OP' be obtained from OP by a rotation through a right angle in a definite sense, the pencils $O[P]$, $O[P']$ are equal, and therefore projective (Art. 24).

But, if we now take OP' as OQ and find the corresponding ray OQ', it will be in the same straight line as OP. Thus the elements of the projective pencils $O[P]$, $O[P']$ correspond doubly, and they form an involution pencil. Such an involution is clearly elliptic, for each double ray must be at right angles to itself, a condition which cannot be satisfied by any real lines.

Because a rectangular involution is elliptic, it has real common mates with any concentric involution pencil (Art. 107). Thus in every involution flat pencil which is not rectangular there exists one real pair of mates at right angles.

There can, however, be one such pair only, for if two pairs of mates of an involution are rectangular, the involution to which they belong is altogether rectangular, since it is uniquely determined by the two pairs of mates, and the rectangular involution clearly satisfies the requirements.

An elliptic involution range on a straight line x can be projected, from a point outside its base, by a rectangular involution pencil. For, let (P, P') (Q, Q') be two pairs of mates, which determine the involution range. Describe circles on PP', QQ' as diameters. Since the involution is elliptic, P, P' are separated by Q, Q' and the circles intersect at real points C, D, symmetrically situated with regard to x. The pencil obtained by joining C to the pairs of the involution range on x is an involution pencil, in which (CP, CP') and (CQ, CQ') are pairs of mates. But, from the property of the angle in a semi-circle, CP and CP' are perpendicular, and so also are CQ and CQ'. Thus the involution pencil through C is rectangular. The same applies if we project from D.

If the original involution is hyperbolic, the circles on PP', QQ' as diameters do not intersect in real points. Clearly no hyperbolic involution range can be incident with any elliptic involution pencil, and, in particular, it cannot be incident with a rectangular involution.

110. The Frégier point. An involution flat pencil whose vertex is on the conic determines an involution of points on the conic. In particular, if the involution pencil be rectangular, we reach the following theorem. If O be any point on a conic, OP, OP' two perpendicular chords, meeting the conic at P, P' respectively,

PP' passes through a fixed point F. Taking P coincident with O, OP, OP' are the tangent and normal at O and PP' coincides with the normal at O. The fixed point F therefore lies on the normal at O. The point is called the **Frégier point** from its discoverer.

If the conic be a rectangular hyperbola and OP, OP' be drawn parallel to its asymptotes, PP', and therefore the Frégier point, is at infinity. In any other position, therefore, PP' is parallel to the normal at O. Thus if on any chord PP' of a rectangular hyperbola as diameter, a circle be constructed meeting the curve at O, O' the normals at O, O' are parallel to PP'.

Examples

1. Show that in any conic if G, G' be the points where the normal at P meet the axes, F the Frégier point corresponding to P, then P, F are harmonically conjugate with regard to G, G'.

2. Show that in a parabola the locus of the Frégier point is another parabola, equal to the given one.

3. Given two points A, B on a conic, find two other points P, Q on the conic such that A and B shall lie on a circle of which PQ is a diameter.

[P, Q are the intersections with the conic of the line joining the Frégier points corresponding to A and B.]

111. Involutions of conjugate elements with regard to a conic. The two collinear projective ranges formed by associating with each point of a line its conjugate point with regard to a conic (Art. 52) define an involution, since, from the symmetry of the conjugate relation, two corresponding points correspond to each other doubly. The double points of this involution are the points where the straight line meets the conic.

Similarly conjugate lines through a point form an involution of which the double rays are the tangents from the point.

In particular conjugate diameters form an involution, of which the double rays are the asymptotes.

Since the involution of conjugate diameters has one real pair of mates at right angles and one only, we obtain a new proof of the theorem of Art. 60 that a conic has one, and only one, pair of axes.

Example

Show that two concentric conics have one pair of common conjugate diameters and that these are always real if one of the conics is an ellipse.

The tangent at P to a conic meets a concentric conic at Q, R. Show how to find P so that QR shall be bisected at P.

112. Radical axis of two circles. Let there be two circles in a plane, with centres A and B (Fig. 43), and radii a, b respectively.

From a point P of the plane let any lines PQR, PST be drawn, meeting the circles at Q, R and S, T respectively.

Consider the locus of P if $PQ.PR = PS.PT$.

Let PX be the perpendicular from P on AB, and let O be the middle point of AB.

By a well-known property of the circle

$$PQ.PR = AP^2 - a^2 = AX^2 + XP^2 - a^2,$$
$$PS.PT = BP^2 - b^2 = BX^2 + XP^2 - b^2,$$

so that, if $PQ.PR = PS.PT$,

$$AX^2 - BX^2 = a^2 - b^2,$$
$$(AX - BX)(AX + BX) = a^2 - b^2,$$

or
$$2.AB.OX = a^2 - b^2.$$

Thus OX is constant, X is a fixed point of AB, and the locus of P is a fixed straight line, namely the perpendicular through X to AB.

This locus is termed the **radical axis** of the two circles. Since $PQ.PR = PU^2$, and $PS.PT = PV^2$, where PU, PV are tangents from P to the circles, we have $PU = PV$, so that tangents to two circles from any point of their radical axis external to the circles are equal.

If the two circles meet at real points C, D, the tangents from C (or D) to both circles are zero, and therefore equal, so that C, D are points on the radical axis, which is then the common chord of the circles.

An important limiting case arises when one of the circles is a **point-circle,** that is, a circle of zero radius. In this case there is still a radical axis. If L (Fig. 43) is such a point-circle, PL is the tangent to L from P, and the condition for P to be on the radical axis of L and the circle centre A is that $PL^2 = PU^2 = PQ.PR$. Since PL^2 is here essentially positive, the radical axis then lies entirely outside the circle centre A.

113. Coaxal circles. A set of circles which are such that all pairs of the set have a common radical axis are termed **coaxal circles.**

The centres of such circles all lie on a fixed line. For let PX (Fig. 43) be the given radical axis, A the centre of a given circle of the set. Then, if B is the centre of any other circle of the set, AB is perpendicular to PX and is fixed, since A and PX are given.

If the radical axis meets one of the circles of the set at C, D, then CD must be the common chord of this circle with every other circle of the set, and the coaxal circles form a system of circles passing through two fixed points. If, on the other hand, the radical axis does not meet any circle of the set in real points, then no circle of the set intersects any other, and the radical axis is external to every circle of the set.

By Art. 112 the tangents from a point P to every circle of a coaxal system are equal. If, with P as centre, and radius equal to the tangent PU from P to any given circle of the system (that with centre A in Fig. 43), a circle be described, this circle passes through all the points of contact of tangents from P to the circles of the system and therefore cuts all these circles orthogonally. Such

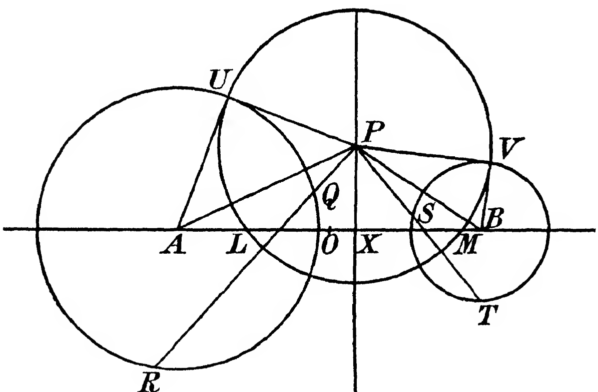


FIG. 43.

circles, centre P, therefore form a system orthogonal to the given system of coaxal circles.

This orthogonal system of circles is itself a coaxal system. For if A is the centre of the circle k_1 of the original system, the tangent from A to any orthogonal circle is the radius AU of the circle k_1. All such tangents from A to circles of the orthogonal system are equal. Similarly, if B is the centre of a circle k_2 of the original system, the tangents from B to circles of the orthogonal system are equal to the radius of k_2. Thus AB is the common radical axis of circles of the orthogonal system.

Consider now the case where a circle of the original system reduces to a point-circle, say L. Clearly L must lie on the line of centres. Also XL is equal to the length of the tangent from X to any circle of the original system. Hence L lies on a circle centre X, whose radius is equal to the length of the above tangent. This circle, when it is real, meets AB at two points L, M symmetrically situated with respect to X. These are called the **limiting points** of the original set of coaxal circles.

The limiting points are real if X is outside the circles of the original system, that is, if these circles have no real intersections. In this case the circles of the orthogonal system all pass through L, M, and form a set of coaxal circles with real intersections.

If, however, the circles of the original system have real intersections C and D, X is the middle point of CD and internal to every one of these circles. No real tangent from X to these circles can be drawn, and there are no real points L, M.

In the latter case the common radical axis AB of the orthogonal system does not meet the circles of this system in real points. Further, if we take P at C, or D, the corresponding length of tangent to the circles of the original system (which pass through C and D) is zero, so that C, D are point circles, that is, limiting points, of the orthogonal system.

Thus **two orthogonal sets of coaxal circles are such that: (i) their lines of centres are at right angles; (ii) one only has real intersections, which are real limiting points of the other.**

114. Coaxal circles determine an involution on any straight line. Let y be the common radical axis of a system of coaxal circles. Let x be any straight line, meeting y at O and any two circles of the system at P, P', Q, Q'.

Then, by the defining property of coaxal circles, since O is on the radical axis

$$OP.OP' = OQ.OQ'.$$

Accordingly the product $OP.OP'$ is constant, and the points P, P' are mates in an involution range on x, whose centre is O (Art. 99).

If O is outside the circles this product is positive, and the involution is hyperbolic; its double points are then the points of contact of the circles of the system which touch x.

If O is inside the circles, the product is negative and the involution is elliptic and has no real double points.

Examples

1. Prove that the radical axes of a given circle, not belonging to a given coaxal system, with the circles of this coaxal system are concurrent.

2. Two systems of coaxal circles have the same radical axis, and O_1, O_2

are the limiting points of the two systems on one side of this axis. A circle c_1 of the first system passes through O_2, and a circle c_2 of the second system passes through O_1. If O_1O_2 meet the circles c_1, c_2 again at Q_1, Q_2 and the common radical axis at P, prove that

$$PQ_1.PQ_2 = PO_1.PO_2.$$

3. If a set of circles be drawn, each passing through a pair of mates of an involution on a straight line, the radical axes of these circles taken in pairs all pass through one fixed point.

4. If two straight lines meet three circles in three pairs of points of an involution, the three circles have, in general, a common radical axis. Discuss the case of exception.

5. If coaxal circles have real intersections, prove that these intersections are the points from which the involution determined on the line of centres by the coaxal circles is rectangularly projected.

6. Prove the following construction for the centre O and double points A, B of an involution on a straight line x, given by two pairs of mates (P, P'), (Q, Q'). Describe any circles through P, P' and Q, Q' respectively intersecting at C, D. Then CD meets x at O. If OT is the tangent from O to either circle, then the circle centre O and radius OT meets x at A, B.

7. Prove the following construction for the common mates of two collinear hyperbolic involutions.

Let (A_1, B_1) be the double points of one involution, (A_2, B_2) those of the other. Let any circles passing through A_1, B_1 and A_2, B_2 respectively meet at L, M. Then the points of contact of the circles through L, M touching the common base of the two involutions are the common mates required.

8. Prove the following construction for the common mates of two collinear elliptic involutions.

Let C_1, D_1 and C_2, D_2 be the points (symmetrically situated with respect to the common base) from which the given involutions can be rectangularly projected. Then a circle can be described through C_1, D_1, C_2, D_2 and this circle meets the common base at the required points.

9. Prove the following construction for the common mates of an elliptic and a hyperbolic involution in the same straight line x.

Let A_1, B_1 be the double points of the hyperbolic involution, C_2, D_2 the points from which the elliptic involution can be rectangularly projected.

Describe a circle touching x at A_1 (or B_1) and passing through C_2 (or D_2). Let O be the middle point of A_1B_1 : join OC_2 meeting the circle at E. Then the circle EC_2D_2 meets x at the required points.

EXAMPLES VIIA

1. Prove that, if (P, P') are mates in an involution, and (P, P'') are mates in a cobasal involution, the forms $[P']$, $[P'']$ are projective, and show that their self-corresponding elements are the common mates of the two given involutions.

2. If (P, P_1) are mates in an involution ϖ_1, (P_1, P_2) are mates in another (cobasal) involution ϖ_2 and (P_2, P_3) are mates in a third cobasal involution ϖ_3, show that $[P]$ is projective with $[P_3]$, and that, if ϖ_1, ϖ_2, ϖ_3 have a pair of mates A, A' in common, then (P, P_3) are mates in a fourth involution ϖ_4. Show also that the four pairs of double points of ϖ_1, ϖ_2, ϖ_3, ϖ_4 are pairs of mates in a fifth involution.

3. (A, A') and (B, B') are pairs of mates in an involution ϖ whose double elements are C, D. Prove that C, D are mates (i) in the involution ϖ_1, of which (A, B) and (A', B') are pairs, and (ii) in the involution ϖ_2 of which (A, B') and (A', B) are pairs.

If P, Q are mates in ϖ_1, and Q, R are mates in ϖ_2, prove that P, R are mates in ϖ.

4. A range $[P]$ on a given line l is projected from two different vertices U, V into ranges $[P_1]$, $[P_2]$ on a second straight line l'. Prove that the necessary and sufficient condition that $[P_1]$, $[P_2]$ should form an involution is that UV should be harmonically divided by l, l'.

5. State, and prove independently, the theorem obtained from Ex. 4 above by reciprocation.

6. If (A_1, A_2), (B_1, B_2), (C_1, C_2) be three pairs of points of an involution on a straight line, show that

$$\frac{C_1A_1}{C_2A_2}B_1B_2+\frac{C_1B_1}{C_2B_2}B_2A_1+\frac{C_1B_2}{C_2B_1}A_1B_1=0.$$

7. Prove that any elliptic involution pencil can be projected into a rectangular involution by a real projection.

8. Prove that if a straight line l meet two coplanar projective pencils of vertices A, B in an involution, l and AB must be conjugate with respect to the product of the pencils.

9. If (A, A') be a fixed pair of mates in an involution on a straight line, (P, P') any other pair of mates, prove that $\left(\frac{AP.AP'}{A'P.A'P'}\right)=$ constant, and find the value of this constant in terms of the distances of A, A' from the centre of the involution.

10. The sides BC, CA, AB of a triangle ABC meet a straight line at P, Q, R. If P', Q', R' are mates of P, Q, R in an involution, prove that $P'A$, $Q'B$, $R'C$ are concurrent.

11. If through the vertices of one triangle lines a_1, b_1, c_1 be drawn parallel to the sides of another triangle, and through the vertices of the latter triangle lines a_2, b_2, c_2 be drawn parallel to the sides of the first triangle; prove that if a_1, b_1, c_1 are concurrent, so are a_2, b_2, c_2.

12. Show that the tangents at the points of an involution on a conic form an involution of tangents having the same axis and centre as the given involution of points.

13. The rays p, p' are mates in an involution pencil whose double rays are a, b; a conic s meets p, p' at P_1, P_2; P_1', P_2' respectively. Show that each of the lines P_1P_1', P_1P_2', P_2P_1', P_2P_2' meets a and b in points which are conjugate for s. Hence prove that these four lines touch a fixed conic k, which is independent of the choice of the particular pair of mates p, p', and which touches a, b, and the four tangents to s at its intersections with a and b.

14. State and prove the reciprocal of the result of Ex. 13.

15. Given two pairs of conjugate diameters of a conic in position (but not in length), show how to construct the axes of the conic in position.

16. Show that if a simple quadrilateral exist which is inscribed in a conic s and circumscribed to a conic s', there exist an infinite number of such simple quadrilaterals and they have the same intersection of diagonals.

[Use the result of Ex. 13.]

17. O, U are conjugate points with regard to a conic. Through O a ray OPQ is drawn meeting the conic at P, Q. Prove that UP, UQ are mates in an involution.

18. If a chord QQ' of a parabola meet a diameter PV at O, and if QV, $Q'V'$ be ordinates to this diameter, prove that $PV.PV'=PO^2$.

19. If from a point T outside a parabola a tangent TP and a chord TQQ' be drawn, and if the diameter through P meet QQ' at K, show that $TQ.TQ'=TK^2$.

20. Two coplanar elliptic involution pencils have different vertices A, B. Show that through any point P of their plane two conics can be drawn passing through A and B, on either of which the two involution pencils determine the same involution.

By taking P on AB prove that there are two real straight lines, on each of which the given pencils determine the same involution. [Use Art. 108.]

21. A, B are the vertices of two coplanar involution pencils with real double rays (a_1, a_2); (b_1, b_2) respectively. If $a_1b_1=C$, $a_2b_2=D$, $a_1b_2=E$, $a_2b_1=F$, prove that the two pencils determine the same involution on any conic through A, B, C, D; or through A, B, E, F.

22. Two involution ranges on the same conic have a double point A in common. If V is the point where the tangent at A meets the join of their other double points, (P, P') any pair of mates in one involution, (Q, Q') the points where VP, VP' respectively meet the conic again, prove that Q, Q' are mates in the other involution.

Discuss the particular case of this theorem when the conic is a line-pair, the involutions lying on different lines of the pair.

23. A pair of mates in a rectangular involution of vertex O meets a fixed line l at P, Q, and P', Q' are mates of P, Q respectively in a given involution upon l. Through P', Q' parallels are drawn to OQ, OP respectively, meeting at R. Show that the locus of R is a straight line.

24. If F is the Frégier point of P, and C the centre of the conic, prove that CP, CF are equally inclined to the axes.

25. If the tangent at P to a hyperbola whose centre is C meet the asymptotes at Q and R, prove that the Frégier point of P is the intersection of the tangents at Q and R to the circle through C, Q and R. [Use Ex. 24.]

26. If AB, CD are two perpendicular chords through the Frégier point F of P, and the circles PAB, PCD meet the conic again at R and S respectively, prove that RS passes through F.

EXAMPLES VIIB

1. A, B, C are three points in order on a straight line, where $AB=3''$, $BC=2''$. Construct the point D which is the harmonic conjugate of C with respect to A and B, and the points X, Y which are the common harmonic conjugates of the pairs (A, D) (B, C).

Find also the points which are the common harmonic conjugates of the pairs (A, C), (B, D).

Are there any common harmonic conjugates of the pairs (A, B), (C, D)?

2. O, P, Q are points on a straight line; $OP=2{\cdot}5''$, $OQ=4''$. P, Q are mates in an involution on the line, of which O is a double point. Construct the other double point, and the mate of the point R, where $OR=6''$.

3. Two circles have radii 1·5″ and 1″ respectively, and their centres are 3·5″ apart. Construct graphically: (1) their radical axis; (2) the limiting points of the system of coaxal circles defined by the given circles; (3) the circle of this system passing through the point on the radius of the circle of radius 1″ perpendicular to the line of centres, and at a distance of 2·5″ from the centre of the last-named circle.

4. (OP, OP'), (OQ, OQ') are mates in an involution pencil. The angles POQ, QOQ', $Q'OP'$ (measured in the standard sense) are 45°, 30°, 60° respectively. Construct the double rays of the involution.

5. Draw two circles of radii 2 inches and 1 inch respectively, with their centres 4 inches apart, and mark a point A distant 4 inches from both centres. Draw the circle which passes through A and cuts the first two circles orthogonally.

6. The centre of a conic is at the origin, and the conic passes through the points (1, 0) (0, 2) and (−1, 2). Construct its axes in position, without drawing the curve.

7. P, Q, Q', P' are four points in order on a straight line. $PQ=3$ cm., $QQ'=2$ cm., $Q'P'=5$ cm. Construct the common mates of the involutions defined by the pairs (P, P') (Q, Q') and (P, Q') (Q, P') respectively.

8. Two conjugate semi-diameters of an ellipse are of lengths 2″ and 3″ respectively, and make an angle of 60° with one another. A hyperbola has these diameters for asymptotes. Construct the common conjugate diameters of the ellipse and hyperbola in position.

CHAPTER VIII

FOCI AND FOCAL PROPERTIES OF THE CONIC

115. Foci of a conic. A **focus** of a conic s is defined to be a point such that conjugate lines through it are perpendicular.

There can be no focus which does not lie on an axis of the conic. For, let S be a focus, and x the diameter through S. Then, by the property of the focus, the line through S conjugate to x is perpendicular to x, so that the direction conjugate to the diameter x is at right angles to x; therefore x is an axis.

Let now U^∞, V^∞ (Fig. 44) be two rectangular points at infinity which are not conjugate points for s. This requires (i) that s is not a circle; (ii) that neither U^∞ nor V^∞ lies on an axis.

Consider the pencils of lines u, u' conjugate for s, through U^∞, V^∞. Then, by Art. 52,

$$[u] \barwedge [u'].$$

Thus the product of $[u]$, $[u']$ is a conic k passing through U^∞, V^∞.

Now every focus S must lie on k. For, if we join SU^∞, SV^∞ these, being perpendicular lines through a focus, are conjugate for s, and their intersection is therefore a point of k. Further, every intersection of k with an axis x is a focus. For the involution of conjugate lines through S has then two pairs of rectangular mates, namely x and the perpendicular to x through S, and the pair SU^∞, SV^∞. All the possible foci are therefore given by the intersections of k with the axes.

If the original conic s is a central conic, the line at infinity does not touch s, so that $U^\infty V^\infty$ is not self-conjugate and the pencils $[u]$, $[u']$ are not perspective; k is then a non-degenerate conic, which is clearly a rectangular hyperbola, since it passes through U^∞, V^∞. Further the tangents to this hyperbola at U^∞, V^∞ are the lines through these points conjugate to $U^\infty V^\infty$, that is CU^∞, CV^∞, where C is the centre of s. C is accordingly the meet of the asymptotes of k, and therefore the centre of k, as well as of s. If now x, y are the axes of s, they are perpendicular diameters of k. Hence, by Art. 83, one only, say x, meets k at two real points S, H.

These are the only real foci of s and are necessarily symmetrical with respect to C.

The axis on which the foci lie is called the **focal axis,** the other being the **non-focal axis.**

Since the involution of conjugate lines through a focus, being rectangular, has no real double lines, no real tangents can be drawn from a focus to the conic, and foci are always points internal to the conic. Thus the line joining them meets the conic at real points A, A', the centre C being the common mid-point of AA', SH.

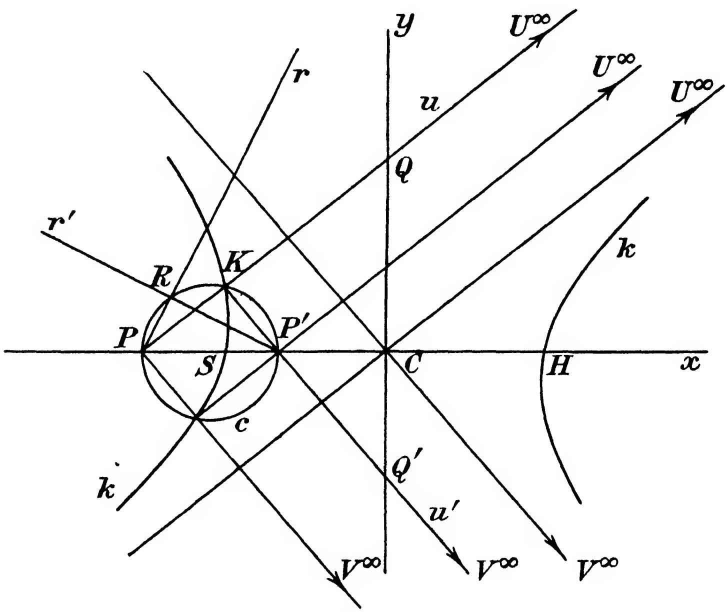


FIG. 44.

In the ellipse, the line at infinity does not meet the curve in real points, so that its pole C is a point internal to the conic. The segment AA' is therefore internal to the conic and S, H lie between the vertices A, A'. That the focal axis is in this case the major axis will be proved later (Art. 119).

In the hyperbola, the line at infinity meets the curve in real points, and C is external to the conic. The segment AA' is therefore external to the conic, and S, H lie outside the vertices, A, A'. The focal axis is here necessarily the transverse axis, since the conjugate axis does not meet the curve in real points.

If the conic s is a parabola, $U^\infty V^\infty$ touches s and is self-conjugate; it is therefore a self-corresponding ray of the pencils $[u]$, $[u']$, which are accordingly perspective; k therefore breaks up into the line at infinity and another straight line l.

But the point of contact L of the accessible tangent from U^∞ to s, being the pole of this tangent, lies on the conjugate line through V^∞ and so is a point of k. Similarly the point of contact M of the accessible tangent from V^∞ to s is a point of k. These two points, being accessible, must lie on the component l of k. Also L, M cannot lie on the axis, since the tangents at L, M are oblique to the axis. Hence l cannot be identical with x, and must meet it at some point S. Moreover, S must be an accessible point. For if S were at infinity, it must be the point X^∞ at infinity on x; but X^∞ is the point of contact of one tangent from U^∞ or V^∞, namely the line at infinity. Also LX^∞ is the polar of U^∞, and MX^∞ is the polar of V^∞, with respect to s. If therefore L, M, X^∞ all lie on l, l must be the polar of both U^∞ and V^∞ with respect to s, or two different points would have the same polar with respect to a non-degenerate conic, which is impossible.

Thus here again the locus k meets the axis of the parabola at two real points, one S at a finite distance, which is usually referred to as *the* focus of the curve, and one H^∞ at infinity.

If the conic s is a circle, every pair of rectangular points U^∞, V^∞ are conjugate, and the above argument does not apply. In this case all conjugate diameters are perpendicular, so that the centre C is a focus. Also no other point P can be a focus. For if we draw any line q through P, its pole Q lies on the perpendicular n through C to q. PQ is the line through P conjugate to q, and this cannot be perpendicular to q unless Q is at infinity on n, that is, q passes through C. Thus the only rectangular conjugate lines through P are the diameter through P and its perpendicular.

Examples

1. Prove that the absolute length of the semi-diameter of k along the axis y is CS, and that, if the conic s is a central conic, the conic k and its conjugate hyperbola k' together pass through the corners of a fixed square.

2. In the case where the given conic is a parabola, show that the line joining the points of contact of two perpendicular tangents passes through the focus.

116. Involution of orthogonal points on an axis. Let a pair of conjugate lines u, u' through U^∞, V^∞ (Fig. 44) meet an axis x at P, P'. The points P, P' are in general distinct, unless they happen to coincide with a focus. We shall further suppose that neither of them is at infinity.

The points P, P' cannot be conjugate for s. For, if they were, the polar p of P would be the line through P' perpendicular to x. This is conjugate to every line through P, in particular to u; but, being perpendicular to an axis, it cannot pass through V^∞, so that u' cannot pass through the point conjugate to P on x.

On PP' as diameter describe a circle c (Fig. 44). This must pass through the intersection K of u, u', since these are perpendicular lines through P, P' respectively.

Consider now the pencils of conjugate lines r, r' through P, P' respectively; then $[r]\barwedge[r']$ by Art. 52. Hence the meet R of r, r' describes a conic t passing through P, P'.

When r coincides with u, r' coincides with u', and R with K. Thus the conic t passes through K.

When r is along x, r' passes through the pole of x, that is, through the point at infinity on y. The tangent at P' to t is therefore parallel to y. Similarly the tangent at P to t is parallel to y.

Thus the circle c and the conic t have in common three points P, K, P' and the tangents at P, P'. Hence they coincide altogether.

Accordingly all points R lie on the circle, and all conjugate lines r, r' are perpendicular.

The points P, P' have therefore the property that conjugate lines through P, P' are perpendicular.

The points Q, Q' at which u, u' meet y can be shown in like manner to have the same property.

Pairs of points which have this property will be referred to as **orthogonal** points with respect to the conic.

If we now take P at infinity, and remember that $V^\infty C$ is the ray conjugate to the line at infinity through U^∞, it will follow that P' is at C. In this case every line through P' is conjugate to the line at infinity through P^∞, but since the line at infinity may be regarded as perpendicular to every direction, it may be regarded as perpendicular to all the lines through P', which are conjugate to itself. On the other hand, accessible lines through P^∞ are parallel to x and so are conjugate to the other axis y through P', and perpendicular to it.

Thus C and the point at infinity on x still possess the essential property of orthogonal points. It should be noticed that this pair form an exception to the rule, that orthogonal points are not conjugate for s.

Since the ranges $[P]$, $[P']$ are the intersections of x with the projective pencils $[u]$, $[u']$, they form projective ranges. But, further, if we join $P'U^\infty$, PV^∞ these lines are perpendicular and, therefore, by the property of orthogonal points, conjugate. Hence if we take u along $U^\infty P'$, u' is along $V^\infty P$, so that if P is at P', then P' is at P. The pair P, P' therefore correspond doubly and the orthogonal points on an axis form an involution, of which the centre C of the conic is the centre.

In the above, x may be either axis. But the double points of these involutions must be an intersection with x of the conic k of Art. 115. The double points on the focal axis are therefore the

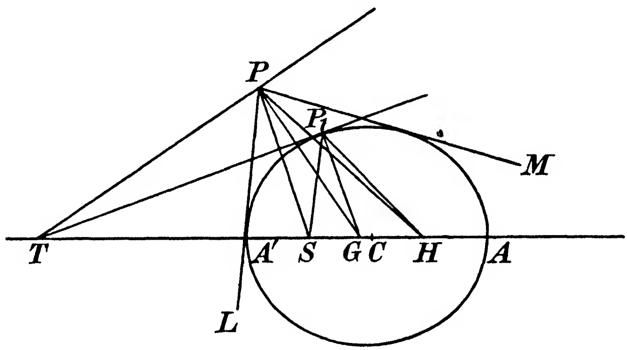


FIG. 45.

foci S, H, and the involution on this axis is hyperbolic, and we have

$$CP.CP' = CS^2 = CH^2.$$

On the non-focal axis there are no real intersections with k, and therefore no real double points of the involution of orthogonal points on that axis, which involution is therefore elliptic.

In the case of the parabola, where there is only one accessible axis, one double point H is at infinity and the other S is at a finite distance. Since these divide PP' harmonically, orthogonal points P, P' are symmetrically situated with respect to the focus.

117. The bisectors of the angles between the focal distances. Let P (Fig. 45) be any point, not a focus, PG, PT the rectangular mates of the involution of conjugate lines through P meeting an axis at G, T. Then G, T are orthogonal points for the conic. Hence

$$CG.CT = \text{constant},$$

for either axis.

If the axis in question be the focal axis, we have further

$$CG.CT = CS^2 = CH^2.$$

Moreover, G, T are harmonic with respect to the foci S, H; hence PG, PT are harmonic with respect to PS, PH, and being perpendicular, are the bisectors of the angle SPH.

If now PL, PM are the tangents from P to the conic, PG, PT, being conjugate lines, are also harmonic with respect to PL, PM (Art. 49) and are therefore also the bisectors of the angle between PL, PM.

Hence **two rectangular conjugate lines through P are the common bisectors of the angles between** (i) **the focal distances,** (ii) **the two tangents from P to the conic.**

If P is on the conic at P_1, the tangent at P_1 and the line through P_1 perpendicular to the tangent, which is known as the **normal** at P_1, are clearly conjugate lines through P_1 and also perpendicular.

Thus: (i) **if the normal and tangent meet the focal axis at G, T, then** $CG.CT = CS^2$, and (ii) **the tangent and normal at any point on the conic bisect the angles between the focal distances.**

If P_1T is here the tangent, then, since a real tangent to a conic necessarily lies outside the curve, T lies outside AA' (and therefore outside SH) in the case of the ellipse, and inside AA' (and therefore inside SH) in the case of the hyperbola.

The internal bisector of the angle SP_1H is therefore the normal when the conic is an ellipse, and the tangent when it is a hyperbola.

In the parabola, H is at infinity on the axis. Hence the tangent and normal bisect the angles between the axis and the focal distance; and for a point P not on the parabola the two tangents from P, and the diameter and focal distance, have the two rectangular conjugates through P as common bisectors. Also, since H is at infinity, S becomes the middle point of GT.

Examples

1. If the tangent and normal at P meet the non-focal axis at T' and G', prove that S, H, P, T', G' are concyclic and $CG'.CT' = -CS^2$.

2. Show that two parabolas which have a common focus and their axes in opposite directions intersect at right angles.

3. PQ is a focal chord of a parabola, and the normal at P meets the curve again at R; if QR meets a line through P parallel to the axis at V, and U is the midpoint of PV, prove that UQ is the tangent at Q.

4. Prove that through any point P of the plane, two conics can be drawn having given foci, and that, of these two, one is an ellipse, and the other a hyperbola, and they meet at right angles at all their intersections.

5. Show that two parabolas with a given focus and axis pass through a given point P, and that they intersect at right angles.

6. Show that the two tangents from P to any conic are equally inclined to the tangent and normal at P to a confocal conic through P.

7. Having given three tangents to a conic and a focus determine the other focus and the points of contact of the tangents.

118. The eccentricity. The polar of a focus is called a **directrix.** Since a focus lies on an axis, its directrix is perpendicular to that axis. Also, from the symmetry of the curve

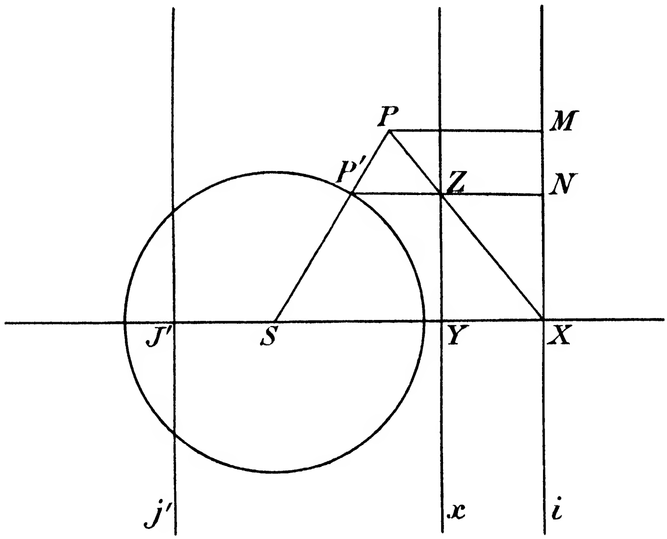


FIG. 46.

with regard to the centre, and since the foci are symmetrical points, the directrices are symmetrically situated with regard to the centre.

Let S (Fig. 46) be a focus of a conic s, XM the corresponding directrix, X being the foot of the perpendicular from S on this directrix. Construct the figure in plane perspective with s, when S is taken as pole of perspective, and the directrix as the vanishing line i for the conic, the axis of perspective YZ being any arbitrary line x parallel to the directrix.

The figure corresponding to s will be another conic s', in which the point corresponding to S and the line corresponding to i are pole and polar. But S is self-corresponding and i corresponds to the line at infinity i'^{∞}. Hence S is the centre of s'.

Further conjugate lines through S, which are at right angles since S is a focus, transform into conjugate diameters of s'. But lines through S transform into themselves by the property of the pole of perspective. Hence all conjugate diameters of s' are perpendicular, that is, every diameter of s' is an axis. Thus s' is a circle (Art. 60).

Let P be any point on s. Join XP, meeting the axis of perspective at Z. Then X'^{∞} is at infinity on SX, and ZX'^{∞} corresponds to ZX, that is, it is the parallel through Z to SX. Since P lies on ZX, P' lies on ZX'^{∞}; but P' also lies on SP, and so is determined by the intersection of SP and the parallel through Z to SX.

Let PM be drawn perpendicular to the directrix, and let $P'Z$ meet the directrix at N.

Since PM, ZN are parallel

$$PM : ZN = PX : ZX.$$

Since $P'Z$, SX are parallel

$$PX : ZX = SP : SP'.$$

Hence $$PM : ZN = SP : SP'$$

or $$SP : PM = SP' : ZN = SP' : YX,$$

Y being the intersection of x and SX.

Since P' describes a circle with S as centre, SP' is constant. Also YX is the perpendicular distance between the axis of perspective and the vanishing line i, and is constant as P varies.

Thus $SP' : YX =$ a constant ratio, which we will denote by e.

Accordingly $SP = e.PM$, or:

The distance of a point on a conic from a focus is in a constant ratio to its distance from the corresponding directrix.

This ratio is called the **eccentricity,** and the symmetry of the two foci and directrices shows that it is the same for either focus, in the case of a central conic.

Let now the vanishing line j' of the circle meet SX at J' (Fig. 46). Then, since by Art. 12 the distance of one vanishing line from the axis of perspective is equal to the distance of the pole of perspective from the other vanishing line,

$$YX = J'S$$

and $$e = SP' : J'S.$$

If j' meets the circle in real points, the conic s is a hyperbola (Art. 34). This requires $SP' > J'S$, so that $e > 1$.

If j' does not meet the circle in real points, the conic s is an ellipse. We then have $SP' < J'S$, so that $e < 1$.

If j' touches the circle, the conic s is a parabola, and $SP' = J'S$, so that $e = 1$.

The property that $SP = e.PM$ will be referred to as the focus and directrix property.

EXAMPLES

1. Two points of a conic being given and also one of the directrices, show that the locus of the corresponding focus is a circle.

2. T is any point on the tangent at P to a conic of which S is one focus, and TR, TU are the perpendiculars drawn from T to SP and to the directrix corresponding to S. Prove that $SR : TU$ is equal to the eccentricity of the conic.

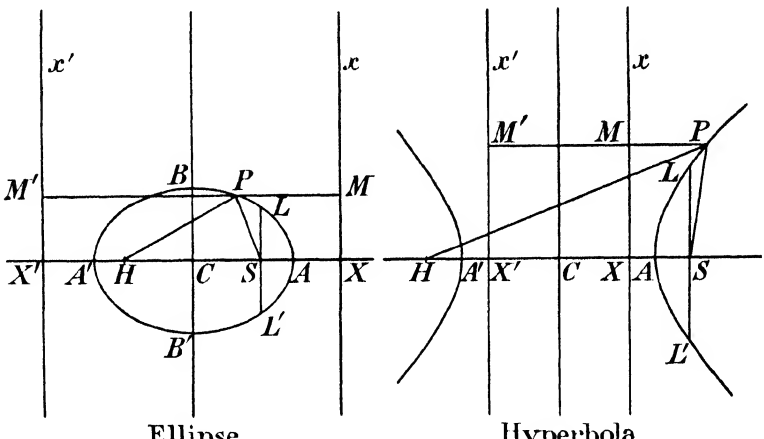


FIG. 47.

119. Relations between fundamental points and lengths. Let Fig. 47 represent the principal points and lines connected with the conic. C is the centre, S, H the foci, x and x' the directrices, meeting the focal axis at X, X'; A, A' are the vertices on the focal axis, B, B' those on the non-focal axis; the latter are shown only in Fig. 47 (a), as they are not real except in the case of the ellipse. The chord LL' through a focus S at right angles to the focal axis is called the **latus rectum.**

From the focus and directrix property:

$$SA = e.AX \quad \text{.............................} (1)$$

$$A'S = e.A'X \quad \text{.............................} (2)$$

where the formulæ hold good for either type of central conic, if signs of segments are taken into account.

From (1) and (2), by addition,

$$A'A = 2e.CX,$$

since C is the middle point of $A'A$.

Hence $$CA = e.CX \qquad\qquad (3)$$

By subtraction of (1) from (2)

$$2.CS = e.A'A,$$

or $$CS = e.CA \qquad\qquad (4)$$

giving the position of the foci.

Note that, in the ellipse, the foci are inside (*cf.* Art. 115), and the directrices outside, $A'A$. These relations are reversed in the hyperbola.

Since the directrices, being polars of internal points, do not meet the conic in real points, it follows that, in the ellipse, the whole curve lies between the directrices; in the hyperbola, the two branches are separated by the directrices; in the parabola, the whole curve lies to one side of the directrix.

Again $$|SL| = e|SX| = |e.SX|$$
$$= |e(CX - CS)| = |CA - e^2CA| = CA|1 - e^2| \qquad\qquad (5)$$

so that $|SL| = CA(1 - e^2)$ in the ellipse, and $CA(e^2 - 1)$ in the hyperbola, CA being taken as positive.

Further, by Art. 76

$$\frac{SL^2}{A'S.SA} = \frac{CB^2}{CA^2}$$

But $$SA = CA - CS = CA(1 - e)$$
$$A'S = A'C + CS = CA(1 + e)$$

Therefore $$CB^2 = \frac{SL^2}{1 - e^2} = \frac{CA^2(1 - e^2)^2}{1 - e^2} = CA^2(1 - e^2) \qquad\qquad (6)$$

Note that $$A'S.SA = CB^2 \qquad\qquad (7)$$

It follows that, in the ellipse, CB is real and less than CA. Thus the focal axis is the major axis of the ellipse. In the hyperbola CB^2 is negative as is to be expected (Art. 76). Denoting by CB_1 its absolute length

$$CB^2 = -CB_1^2$$

and $$CB_1^2 = CA^2(e^2 - 1) \qquad\qquad (8)$$

Note that, in either case, the square of the absolute length of the non-focal axis is

$$CA.SL \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

In the parabola, only S, A, X, L, L' survive as accessible points, and we have

$$SL = SX = 2.SA \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

The latus rectum $LL' = 4.SA$; this is often called the **parameter of the parabola.**

EXAMPLES

1. Prove that a line-pair may be looked upon as the limiting case of a hyperbola when the foci coincide with the centre.

2. Prove that a point-pair may be looked upon as the limiting case of a very flat ellipse or hyperbola, the foci being coincident with the vertices. Show that the eccentricity of a point-pair is unity.

120. The sum and difference of the focal distances. In Fig. 47 let P be any point on the curve, and let the parallel to the focal axis through P meet the directrices at M, M'.

In the ellipse :

$$SP = e.PM,$$
$$HP = e.M'P.$$

By addition

$$SP + HP = e.M'M = e.X'X = 2e.CX = 2.CA,$$

or, **the sum of the focal distances is equal to the major axis.**

In the hyperbola, H being the focus inside the branch remote from P,

$$HP = e.M'P,$$
$$SP = e.MP.$$

By subtraction

$$HP - SP = e.M'M = 2.CA$$

or, **the difference of the focal distances is equal to the transverse axis.**

EXAMPLES

1. The firing of a gun at P is heard at stations A and B at times separated by an interval during which sound would travel a distance l (less than AB). Show how to construct the locus of P.

2. Find the locus of the focus of a parabola passing through two fixed points A, B and the direction of whose axis is given.

121. Tangents from an external point subtend equal or supplementary angles at a focus. Let TP, TQ (Fig. 48) be tangents from T to a conic, P, Q their points of contact, S a focus. Join ST meeting PQ at Z, and let QP meet the directrix corresponding to S at Y. The pole of ST is the meet of the polar of T (that is, PQ) and the polar of S (that is, the directrix). Hence Y is the pole of ST, and Y, Z are harmonically conjugate with regard to P, Q, so that SY, SZ are harmonically conjugate with regard to SP, SQ.

Also SY, ST are conjugate lines through a focus and therefore

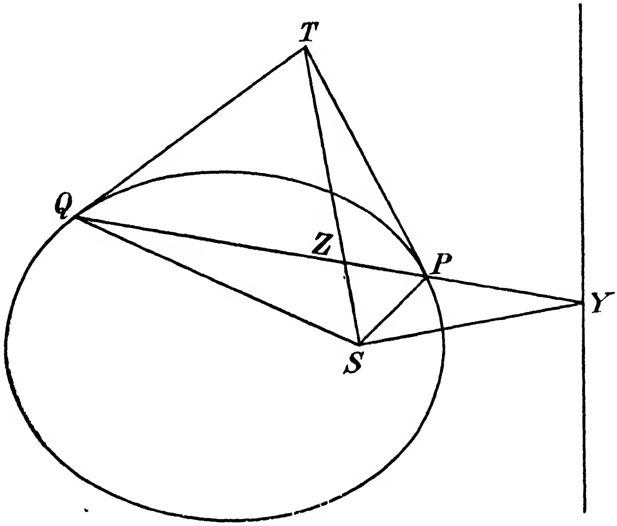


FIG. 48.

perpendicular. Hence (Art. 28) they are the bisectors of the angle PSQ.

The point Y lies outside PQ unless the conic is a hyperbola and P, Q are on different branches (*see* Fig. 47). Thus ST is the internal bisector of the angle PSQ in every case except when P and Q are on different branches of a hyperbola.

Therefore tangents from an external point to a conic subtend equal angles at a focus unless they are tangents to different branches of a hyperbola, when they subtend supplementary angles at a focus.

An important particular case arises when P and Q coincide; T then coincides with them and we obtain the following theorem:

The intercept on a tangent between the point of contact and a directrix subtends a right angle at the corresponding focus.

Examples

1. Prove that tangents to a conic subtend equal or supplementary angles at a focus by considering a plane perspective which transforms the conic into a circle having that focus for centre.

2. Two conics with four real intersections have a common focus. Prove that two of their common chords pass through the intersection of the directrices corresponding to the common focus.

3. Prove the following construction for the pole of any line q with regard to a conic, given the two foci S, S' and the two directrices s, s'. Let q meet s at P, s' at P'. Through S, S' draw perpendiculars to SP, $S'P'$ respectively: these meet at the point Q required.

4. Given a directrix and the corresponding focus of a conic and the direction of one tangent through a given external point O, show how to draw the remaining tangent from O.

122. The rays joining a focus to the intersections of a variable tangent with two fixed tangents describe directly equal pencils. We may state the first theorem of Art. 121 as follows. If TP, TQ are two tangents to a conic, of which S is a focus, twice the rotation (measured positively in a prescribed sense) which brings the line SP into coincidence with the line ST (or ST into coincidence with SQ) will bring the line SP into coincidence with the line SQ. In this form the theorem applies to both cases, whether the angles subtended by TP, TQ at S are equal, or supplementary. We must, however, remember that the coincidences are here irrespective of the sense of segments measured positively on the lines, so that every rotation has an indeterminacy of an integral multiple of two right angles. It follows that we cannot divide by two and say that half the rotation which brings SP to SQ brings SP to ST.

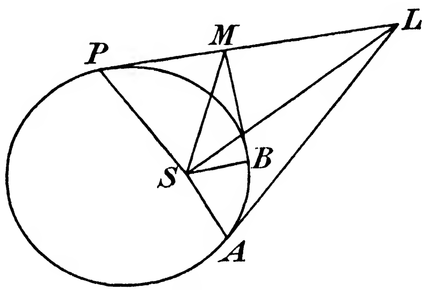


Fig. 49.

Let now A, B (Fig. 49) be two fixed points on the conic, P a variable point, and let the tangent at P meet the tangents at A and B at L and M respectively.

Then, by what has just been stated

$$2(\text{rotation } SL \text{ to } SP) = \text{rotation } SA \text{ to } SP,$$
$$2(\text{rotation } SP \text{ to } SM) = \text{rotation } SP \text{ to } SB.$$

Adding

$$2(\text{rotation } SL \text{ to } SM) = \text{rotation } SA \text{ to } SB,$$

and this must be true for any interpretation of the rotation SL to SM, and some interpretation of the rotation SA to SB.

There must clearly be a finite range of positions of LM over which these interpretations remain the same, for they cannot change abruptly unless one of the points L, M, P passes through infinity.

Throughout this range of positions we have

$$\text{rotation } SL \text{ to } SM = \text{a constant } \alpha, \text{ say.}$$

Consider now two pencils $S[L]$, $S[M]$ of which the second is obtained from the first by this rotation α. The pencils are equal and therefore projective, hence $[L] \barwedge [M]$ and LM touches a conic s, the tangents to which must coincide with those of the given conic over the above-mentioned range of variation. Thus s and the given conic have an infinite number of common tangents and must coincide. Accordingly the same two pencils give the whole of the tangents to the given conic, and we have, for all positions of P, L, M

$$\text{rotation } SL \text{ to } SM = \text{the same constant } \alpha,$$

which proves the theorem required.

The student will find it instructive to trace the relation of the rotation SL to SM to the angle ASB, (i) when the conic is an ellipse lying inside the triangle formed by the tangents, (ii) when the conic is external to the triangle and touches all three sides along the same branch, (iii) when A and B lie on different branches of a hyperbola, (iv) when A, B lie on one branch of a hyperbola and P on the other branch.

EXAMPLE

Prove the converse of the theorem of the present Article, namely that, if SL, SM are corresponding rays of two directly equal concentric pencils through a fixed point S, meeting two fixed straight lines at L, M, then LM envelops a conic of which S is a focus.

123. Focal chords. Let PSQ (Fig. 50) be a focal chord through a focus S, meeting the corresponding directrix at Y. Since S is the pole of the directrix, $\{YQSP\} = -1$, and, by Art. 28

$$\frac{1}{YP} + \frac{1}{YQ} = \frac{2}{YS}.$$

Let PM, QK, SX be the perpendiculars from P, Q, S on the directrix. By similar triangles

$$\begin{aligned} YP : YQ : YS &= PM : QK : SX \\ &= e.PM : eQK : e.SX \\ &= SP : QS : SL \end{aligned}$$

by Arts. 118, 119, LSL' being the latus rectum.

Hence $$\frac{1}{SP}+\frac{1}{QS}=\frac{2}{SL} \qquad\qquad (1)$$

or, **the semi-latus rectum is a harmonic mean between the segments of any focal chord.**

The above will be found to hold good in all cases, even when S is outside PQ, provided the positive sense on the focal chord is taken from the focus towards the *nearer* intersection P with the curve.

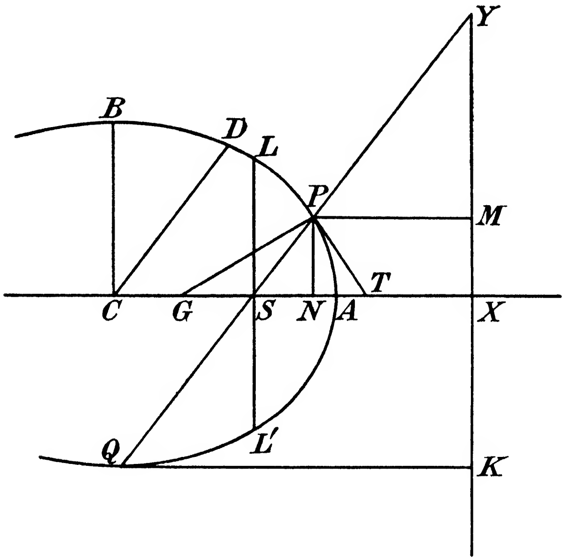


FIG. 50.

Another proposition on focal chords is easily deduced. Newton's Theorem gives at once

$$\frac{QS.SP}{L'S.SL}=\frac{CD^2}{CB^2} \qquad\qquad (2)$$

where CD is the semi-diameter parallel to PQ.

Now, from (1), multiplying up, and remembering that $QS+SP=QP$

$$QP.SL=2.QS.SP=\frac{2.SL^2.CD^2}{CB^2}$$

i.e. $$QP=\frac{2.SL.CD^2}{CB^2}$$

or $$|PQ|=\frac{2.|SL||CD|^2}{|CB|^2}.$$

But $$|CB|^2 = |SL|.|CA|, \text{ by Art. 119 (9)};$$

hence $$|PQ| = \frac{2|CD|^2}{|CA|}$$

or, the lengths of focal chords are proportional to the squares of the absolute lengths of the parallel semi-diameters.

124. Intersection of the normal with the focal axis. Let the normal and tangent at P (Fig. 50) meet the focal axis at G, T, and let PN be the ordinate through P.

We have seen (Art. 117) that

$$CG.CT = CS^2 = e^2.CA^2.$$

But also, PN is the polar of T, since, T being on an axis, its polar is perpendicular to that axis. Thus $\{A'NAT\} = -1$, and

$$CN.CT = CA^2.$$

By division

$$CG = e^2.CN \quad \text{.......................(1)}$$

But $$CN = CX - PM,$$

$$CG = CS - GS,$$

there being here no need to discriminate between the cases of the ellipse and hyperbola, if attention is paid to sign.

Substituting into (1)

$$CS - GS = e^2.CX - e^2.PM$$

But $$e^2.CX = e.CA = CS \text{ (Art. 119)}.$$

Thus $$GS = e^2.PM \quad \text{..................................(2)}$$

or, using Art. 118 $$|SG| = e.|SP| \quad \text{..................................(3)}$$

The distance GN is called the **subnormal.** This also can be expressed in terms of SP, or PM. For

$$GN = CN - CG = CX - PM - CS + GS$$
$$= SX - (1 - e^2)PM \quad \text{..................(4)}$$

These results are often useful.

Examples

1. Prove that the orthogonal projection of the normal PG upon either focal distance is constant and equal to the semi-latus rectum.

[If K = foot of perpendicular from G on SP in Fig. 50, and if the tangent at P meet the directrix corresponding to S at U, prove triangles PKG, USP similar and triangles SKG, UXS similar; and use $SG = e.SP$.]

2. Normals at the extremities of a focal chord PP' meet parallels to the focal axis through P and P' at Q' and Q respectively: show that $PP'QQ'$ is a parallelogram.

125. The feet of focal perpendiculars on a tangent lie on the auxiliary circle. Let SY, HZ (Fig. 51) be perpendiculars from S, H upon the tangent at P. Let HP meet SY at F.

Consider the case where the conic is an ellipse.

The angle $FPY = SPY$, since PT bisects SPH externally by Art. 117. Hence, the angles FYP, SYP being right angles and YP being common, the triangles FYP, SYP are congruent. $\therefore$ $|FP| = |SP|$ and $|FH| = |SP| + |HP| = 2|CA|$. Also C, Y being mid-points of SH, SF respectively, $|CY| = \frac{1}{2}|FH| = |CA|$. Hence Y lies on the auxiliary circle. Similarly Z lies on the auxiliary circle. The proof when the conic is a hyperbola is precisely similar and may be left as an exercise for the student.

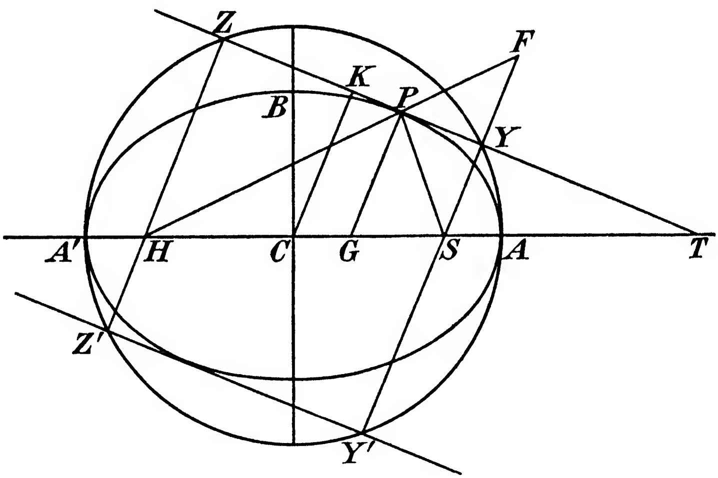


FIG. 51.

If Y', Z' be the feet of perpendiculars upon the tangent parallel to the tangent at P, Y' and Z' also lie on the auxiliary circle.

Also by symmetry

$$|SY'| = |HZ|\,;\quad |HZ'| = |SY|.$$

Hence $SY.HZ = Y'S.SY = A'S.SA$ by the property of segments of chords of a circle.

But $A'S.SA = CB^2$ (Art. 119 (7)), so that

$$SY.HZ = CB^2$$

or, **the product of focal perpendiculars upon a tangent is equal to the square of the non-focal semi-axis.**

If the conic be an ellipse SY, HZ are drawn in the same sense and CB^2 is positive.

If the conic be a hyperbola SY, HZ are drawn in opposite senses and CB^2 is negative and equal to $-CB_1{}^2$ (Art. 119).

Hence $$SY.ZH = CB_1{}^2.$$

Examples

1. Given both foci and a tangent to a conic, show that the conic is uniquely determined, and find the point of contact of the given tangent.

2. A focus and three tangents to a conic are given. Construct the axes of the conic in position and length.

3. If CD be the absolute length of the semi-diameter conjugate to CP, show that
$$SP.HP = CD^2.$$

4. Show that points of contact of tangents from the foci to the auxiliary circle lie on the asymptotes.

5. The focus of a conic slides on a fixed line, the conic itself sliding on a fixed perpendicular line. Find the locus of the centre.

6. A variable line moves so that the product of the perpendiculars upon it from two fixed points is constant. Show that it envelops a conic, of which the two fixed points are foci.

126. The normal PG is inversely proportional to the perpendicular CK from the centre on the tangent at P. For since T, G (Fig. 51) are harmonically conjugate with respect to S, H (Art. 117),
$$\frac{1}{TS} + \frac{1}{TH} = \frac{2}{TG}.$$

But $$TS : TH : TG = SY : HZ : GP,$$
$$\therefore \frac{1}{SY} + \frac{1}{HZ} = \frac{2}{GP},$$
$$\frac{SY + HZ}{SY.HZ} = \frac{2}{GP}.$$

But $SY + HZ = 2CK$ because C is the middle point of SH; also $SY.HZ = CB^2$ by Art. 125,
$$\therefore \frac{2CK}{CB^2} = \frac{2}{GP},$$
or $$CK.GP = CB^2.$$

127. Special properties of the parabola. Many special properties of the parabola are deduced at once from those of the ellipse and the hyperbola by removing one vertex A' and the corresponding focus H to infinity. The line HP then becomes a parallel to the axis. Also e is put equal to unity when it enters into any formula, but $(1-e^2)SA'$ or $2(1-e^2)CA$ is to be put equal to the latus rectum.

It should be noted that, although the *results* may be thus obtained as limiting cases, the *proofs* given for the central conics cannot always be similarly obtained and may require modification for the case of the parabola.

The following are two examples of properties of the parabola deduced as limiting cases of corresponding properties of the central conics.

If, in the last theorem of Art. 124, we write $e=1$, we have at once $GN=SX$, or **the subnormal in the parabola is constant, and equal to the semi-latus rectum.**

The auxiliary circle degenerates into the straight line through A perpendicular to the axis, that is, into the tangent at the vertex. The first theorem of Art. 125 now reads:

The foot of the perpendicular from the focus upon any tangent to a parabola lies on the tangent at the vertex.

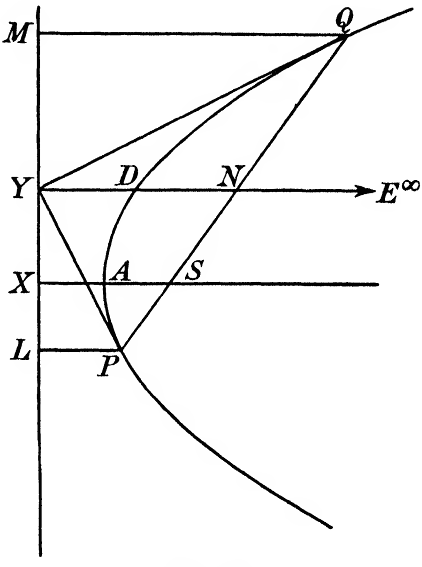


FIG. 52.

Separate proofs of certain other properties of the parabola will now be given.

128. The intersection of perpendicular tangents to a parabola lies on the directrix. Let PSQ (Fig. 52) be a focal chord of a parabola. Let the diameter conjugate to PQ meet the directrix at Y. Then, since the pole of PQ lies on both this diameter and the directrix, it is the point Y. Hence YP, YQ are the tangents at P and Q. Let PL, QM be the perpendiculars from P and Q on the directrix.

By Art. 117 PY bisects LPS and QY bisects MPS. Thus $YPQ+YQP=\frac{1}{2}(LPS+MQS)=$a right angle, since PL and QM

are parallel. The remaining angle PYQ of the triangle PQY must therefore be a right angle, which proves the theorem stated.

129. Parameter of parallel chords of a parabola. We have seen (Art. 76) that if QP (Fig. 52) be a chord of a parabola bisected at N by its conjugate diameter and if this diameter meet the curve at D, then

$$\frac{QN^2}{DN} = \text{the parameter of the chords parallel to } PQ.$$

To find its value take the chord QP to pass through the focus S.

From Fig. 52, since the angle at Y is a right angle, the circle on PQ as diameter passes through Y, its centre being N, the middle point of PQ. Thus $QN = YN$.

Also, since YDN is parallel to the axis, it meets the curve again at E^∞, and, since PQ is the polar of Y, D is harmonically conjugate to E^∞ with regard to Y and N. Hence $DN = YD = \frac{1}{2}YN$.

Accordingly $$\frac{QN^2}{DN} = \frac{4.YD^2}{YD} = 4.YD = 4.SD,$$

by the focus and directrix property.

The parameter is therefore $4.SD$, where D is the extremity of the diameter conjugate to the chords.

130. The circle circumscribing the triangle formed by three tangents to a parabola passes through the focus. If, in the theorem of Art. 122, we take for one of the positions of the variable tangent the line at infinity, which is possible in the case of the parabola, the points L, M become the points at infinity on the other two tangents a and b at A and B respectively. The lines SL, SM are then parallel to a and b, and it follows that, when c varies, the value of the constant angle LSM is one of the angles between a and b.

If now a, b, c are any three tangents, forming a triangle LMN; where N is the intersection of a and b, we have LSM is equal or supplementary to LNM, or the points L, M, N, S are concyclic, and the circle LMN, which circumscribes the triangle formed by the three tangents, passes through the focus S.

Examples

1. TP, TQ are two tangents to a parabola. Show that the circle touching TQ at T and passing through P passes through the focus.

2. Given two tangents to a parabola and their chord of contact, construct the focus and the directrix.

3. PQR being a triangle circumscribed to a parabola, prove that the perpendiculars from P, Q, R to SP, SQ, SR are concurrent.

4. Parabolas are drawn touching the sides of a given triangle; show that the pairs of parabolas, which have their axes perpendicular, have their foci at opposite ends of a diameter of the circumcircle of the triangle.

5. A parabola is given by four tangents. Without finding any point on the curve construct its focus.

6. Show that the four circles circumscribing three of the sides of a complete quadrilateral have one common point.

131. The orthocentre of the triangle formed by three tangents to a parabola lies on the directrix. This is readily proved from Brianchon's Theorem.

Let a, b, c be the three tangents, b', c' the two tangents perpendicular to b, c, and i^{∞} the line at infinity.

Consider the hexagon $abb'i^{\infty}c'c$. Then $(ab, i^{\infty}c')$ (bb', cc') $(b'i^{\infty}, ca)$ are concurrent.

But $(ab, i^{\infty}c')$ is the parallel to c' through ab, *i.e.* the perpendicular from ab on c.

Similarly $(ac, i^{\infty}b')$ is the perpendicular from ac on b.

Hence the Brianchon point is the orthocentre of the triangle.

But (bb', cc') is the directrix, since perpendicular tangents b and b', c and c' meet on the directrix.

The orthocentre therefore lies on the directrix.

EXAMPLE

TP, TQ are two tangents to a parabola; perpendiculars to TP, TQ are drawn through T and P respectively. Show that they intersect on the directrix.

132. Focal chord of curvature of a parabola. The investigation of the circle of curvature (Art. 84) shows that, if the diameter through P meet the circle of curvature at V, then PV is equal to the parameter of the chords conjugate to this diameter.

If the focal radius PS meet the circle of curvature again at U, PU is termed the focal chord of curvature at P. By Art. 117, PU, PV are equally inclined to the normal to the parabola at P, that is to the diameter of the circle through P. Hence, by the symmetry of the circle, $PU = PV =$ parameter of chords parallel to the tangent at P.

But this parameter has been shown (Art. 129) to be $4.SP$. Thus $PU = 4.PS$ and the point U is readily constructed when the focus is known. This gives a point U on the circle of curvature. A perpendicular through U to PU meets the normal at the other extremity of the diameter of the circle through P, and the circle is determined.

Examples

1. If the normal at P to a parabola meet the directrix at H, then the radius of curvature at $P=2HP$.

2. In the plane perspective relation between the parabola and its circle of curvature at P, in which the tangent at P is the axis of perspective, prove that, if C is the centre of the circle and O the pole of perspective, then if OC meet the vanishing line of the circle at K, PK is the diameter of the parabola conjugate to chords parallel to the tangent at P.

3. Using Ex. 2 prove directly that the focal (or diametral) chord of curvature is equal to the parameter of the chords parallel to the tangent at P, without using Art. 84.

EXAMPLES VIIIA

1. Show that the asymptotes of a hyperbola meet the directrices at points on the auxiliary circle.

2. On the transverse axis AB of a hyperbola as diameter a circle is drawn (the auxiliary circle of the hyperbola). A ray through A meets the circle and hyperbola in P, P'. Show that the tangents at P, P' meet on the tangent at B.

3. Having given a focus of an ellipse, a tangent and its point of contact, and one other point on the curve, find the other focus. Show that, when the construction is possible, it leads to two solutions.

4. If TP, TQ be tangents from a point T to a central conic, S, H the foci, show that the bisectors of the angle PTQ meet the non-focal axis in two fixed points when T describes a circle through S, H.

5. Show that if S, H be fixed points and through a point P lines PT, PU be drawn so that the angles SPH, TPU have common bisectors, a conic can be described with S, H as foci to touch PT and PU.

6. If the normal at P meet the non-focal axis at G', show that the projection of PG' upon either focal distance is equal to the focal semi-axis.

7. Prove that if the normal at P to a central conic meet the focal axis at G and the non-focal axis at G', then $PG' : PG = CA^2 : CB^2$.

8. Prove that the pole of the tangent at P to a central conic with regard to the auxiliary circle lies on the ordinate of P.

9. If SY, SZ be perpendiculars from a focus S to tangents TP, TQ, the perpendicular from T to YZ passes through the other focus H.

10. PQ, PR are two focal chords of a conic. Show that QR meets the tangent at P at the pole of the normal at P.

11. PQ, PR are the two focal chords through a point P of a conic. Prove that the pole of QR lies on the normal at P.

12. Prove that the perpendicular drawn through a focus S of a conic to any chord PQ meets the directrix corresponding to S in a point of the diameter conjugate to PQ.

Given a triangle ABC, with an acute angle at C, show that there is only one conic which has AB for a directrix and CA, CB for a pair of conjugate diameters; and show how to determine the auxiliary circle and the points of intersection of this conic with CA and CB.

13. Two parabolas have a focal chord in common and have their axes parallel. Prove that they intersect at right angles. Conversely prove that, if two parabolas have their axes parallel and intersect at right angles at a point, they have a common focal chord which passes through this point.

14. Two parabolas have a common focus. Show that they cannot have more than two real common points, which lie on the internal bisector of that angle between the directrices which contains the common focus.

15. A rectangular piece of paper $ABCD$ is folded so that the corner C falls on the opposite side AB. Show that the crease envelops a parabola of which C is the focus and AB the directrix.

16. The vertex of a constant angle moves on a fixed straight line, while one of its sides passes through a fixed point S. Show that the other side envelops a parabola, of which S is a focus.

17. Prove that a focal chord of a parabola is equal to $4.SP$, where S is the focus, and P is the point where the tangent parallel to the given chord touches the curve.

18. From a point T on the directrix of a parabola tangents TP, TQ are drawn and the chord PQ meets the directrix at K and the diameter through T at R. Prove that

(i) $SP.SQ = SR.SK$,

(ii) $PQ^2 = 4.RS.RK$.

19. If TP, TQ are tangents from a point T to a parabola having S as focus, prove that the triangles SPT, STQ are similar, and that $ST^2 = SQ.SP$.

If tangents from a point O on the axis of a parabola meet any other tangent at points L and M, prove that $SL = SM$ and that S, L, O, M are concyclic.

20. A fixed tangent c to a parabola k is met at P by a variable tangent t, and u is the perpendicular to t through P in the plane of k. Prove that u always touches a second fixed parabola s, whose axis is perpendicular to that of k; and that each of k, s touches c at a point on the directrix of the other.

21. Prove that chords of a conic s which subtend a right angle at a fixed point O not on s envelop a conic of which O is a focus and the polar of O with regard to s is a directrix. [Use Exs. VIIA, 13.]

22. Show that, in any conic, the focal chord of curvature at P is equal to the focal chord of the conic parallel to the tangent at P.

23. S is a focus of a conic s, and SY is the perpendicular from S on the tangent at P. If k is the circle centre S and radius SY, and if Z be an intersection of SP with k, show that the tangents at P, Z to s, k respectively meet on a common chord of k and s, and that the point of contact of the second tangent to s from this intersection lies on SY.

24. Given one focus S of a hyperbola, one asymptote, and a tangent t, show how to construct the other asymptote and focus. Show also how to find the second tangent to the hyperbola from any point of the given tangent t.

25. Show that the hyperbola c, which has foci at two given points, P, Q of an ellipse k and passes through one focus S of k, must also pass through the other focus H, and that the directrix of c corresponding to the focus P meets SH at its intersection G with the normal at P to k.

Show also that, if the normals to k at P, Q meet SH at G, K respectively, the line joining the middle points of PQ, GK is perpendicular to PQ.

26. A variable conic k has one focus at a fixed point S and passes through two other fixed points A and B. Prove that the second focus H describes a

composite locus, composed of an ellipse and hyperbola passing through S and having A, B for foci.

Show that the directrix of k corresponding to the fixed focus S must pass through one or other of two fixed points on the line AB; and that, for any particular position of k, the intersection of this directrix with AB is on the normal at S to that part of the composite locus upon which the other focus H lies.

27. Prove that, if p, q are two fixed perpendicular diameters of a conic k, the meet of conjugate lines with respect to k, which are parallel to p, q, lies on the rectangular hyperbola which has p, q for asymptotes and passes through the foci of k.

EXAMPLES VIIIB

[Except where otherwise stated, the axes of co-ordinates are rectangular.]

1. A right circular cone of semi-vertical angle 60° is cut by a plane inclined at an angle of 15° to the axis of the cone and whose perpendicular distance from the vertex is 4 inches. Construct the asymptotes, foci and vertices of the section.

2. Draw the hyperbola whose directrix is $x=0$, focus $(2, 0)$ and eccentricity $2{\cdot}5$.

3. Draw a triangle SPH with sides $SH=4$ inches, $SP=3$ inches, $HP=$ 2 inches. Draw the tangent at P to the ellipse which passes through P and has S, H for foci; and by geometrical construction find the four vertices of the ellipse, and its directrix corresponding to S.

4. A hyperbola has its vertices at the points $(0, 0)$, $(4, 0)$ and passes through the point P $(5, 4)$. Construct the tangent at P, the foci and the asymptotes.

5. A hyperbola has the lines $y=\pm 1{\cdot}5x$ for asymptotes and passes through the point $(5, 4)$. Construct its foci and vertices.

6. Construct the directrices of the possible conics which have the origin for focus and pass through the three points $(-1, 1)$, $(3, 3)$, $(5, 0)$.

7. The asymptotes of a hyperbola are parallel to the lines $x^2-4y^2=0$. One focus is the point $(3, 0)$ and the semi-latus rectum$=2$. Find the asymptotes in position and draw the curve.

8. A conic has the axis of y for directrix, the point $(1{\cdot}5, 0)$ for corresponding focus and eccentricity$=2$. Draw a chord through the focus which shall be $3{\cdot}5$ units long.

9. Draw a circle of radius $1{\cdot}5$ inches, and mark a point S distant 1 inch from the centre. Construct the polar z of S with respect to the circle, using the ruler and pencil only.

Mark a point Y on the circle (but not on the diameter through S) and find a tangent from Y to the ellipse which has S for a focus, z for corresponding directrix, and the circle for its auxiliary circle.

Find also the points in which this tangent (i) touches the ellipse, (ii) meets a perpendicular tangent to the ellipse.

10. A parabola touches the straight lines

$$y=x+1,\ 2y=x+4,\ 2y+4x+1=0,\ x=0.$$

Construct the focus, axis and vertex of the parabola.

11. The focus of a conic is the origin and the conic passes through the point $(3, 4)$. If the semi-latus rectum$=3{\cdot}5$ and the eccentricity$=\frac{1}{2}$, construct the second foci of the conics which satisfy the conditions.

12. Given the foci S, H of a hyperbola are 4 inches apart and a tangent making an angle of 60° with SH divides SH internally in the ratio 3 : 1, find (i) the point of contact of the given tangent, (ii) the vertices, and (iii) the asymptotes.

13. ABC is a triangle with $BC=4$ inches, $CA=3$ inches, $AB=3{\cdot}5$ inches. Find the foci of the conic which has CA, CB for a pair of conjugate diameters and AB for a directrix; and determine the intersections of this conic with CA and CB.

14. A conic has a focus at the point (2, 0) and touches the lines $x+2y=4$, $2y-x=4$ and $x+y=-3$. Construct the centre, the other focus, the vertices, the directrices and the extremities of the latera recta.

15. A hyperbola touches the co-ordinate axes and the line $x+2y=3$, and has one focus at the point (2, 1). Find its asymptotes, the other focus, and the directrices; and its points of contact with the three given tangents.

16. A conic touches the axis of x at the point (3, 0), and also touches the axis of y; and it has the point (2, 3) for a focus. If the angle between the axes of co-ordinates be 75°, construct it as an envelope.

17. The angle between the positive directions of the axes of co-ordinates being 60°, a conic having a focus at the point (3, 2) touches the axis of y at the point (0, 3) and also touches the axis of x. Construct the point of contact of the x-axis, the second focus, the directrices, the auxiliary circle, the tangents perpendicular to the co-ordinate axes, and the vertices.

18. $ABCD$ is a rectangle, $AB=2$ inches and $BC=1$ inch. A conic touches AD, AC, BC and has one focus at the middle point of AB. Construct the points of contact of AD, AC, BC and the axes of the conic in position and length.

19. A parabola touches two lines AB, AC at B and C, where $AB=5$ cm., $AC=7$ cm., and the angle $BAC=45°$. Find its focus and directrix.

20. A parabola has the axis of y for directrix and touches the line $2y=x$ at the point (4, 2). Find its focus and axis, and the other extremity of the focal chord through (4, 2).

21. A parabola has three-point contact with the circle $x^2+y^2=4$ at the point (2, 0) and touches the line $x-3y+4=0$. Find the point of contact of this line with the parabola, and the axis, focus and directrix of the parabola.

22. An ellipse has the points (0, 0) (6, 0) for the extremities of the major axis, and passes through the point P (1, 1·5). Construct the tangent at P, the foci, and the circle of curvature at P.

23. ACB is a triangle right angled at C, with sides $CB=2$ inches, $CA=4$ inches; and D is the point on AB produced such that the angle $BCD=30°$. The triangle ABC is self-polar with respect to a parabola whose axis is parallel to CD.

Construct (i) the points P, Q, R in which the parabola meets the parallels to the axis through A, B, C respectively; (ii) the focus; (iii) the circle of curvature at P.

CHAPTER IX

IMAGINARY ELEMENTS

133. Point and line co-ordinates in a plane. The position of a point P in a plane may be defined by two co-ordinates x, y given by the intercepts cut off, on two fixed axes, between their intersection or origin and the parallels through P to the axes. In this system of co-ordinates the co-ordinates of the points of any straight line satisfy an equation of the first degree

$$Ax + By + C = 0.$$

If we divide this equation by C it takes the form

$$lx + my + 1 = 0.$$

A straight line is therefore completely defined when we know the two coefficients l, m. These may then be spoken of as the co-ordinates of the line.

The co-ordinates of the points of a curve satisfy a relation which is called the **Cartesian equation** of the curve.

In like manner the co-ordinates of the tangents to a curve satisfy a relation which is called the **tangential equation** of the curve.

If the co-ordinates l, m of a line satisfy a relation of the first degree, this can be put into the form

$$la + mb + 1 = 0,$$

and this shows that the line whose co-ordinates are l, m passes through the point whose co-ordinates are a, b.

An equation of the first degree in l, m is therefore the tangential equation of a point and the lines whose co-ordinates satisfy this equation are rays of a pencil.

If $l = 0$, $m = 0$, x or y or both must be infinite if $lx + my$ is to be equal to the finite quantity -1. Hence $l = 0$, $m = 0$ are the co-ordinates of the line at infinity. Similarly if $x = 0$, $y = 0$ the lines through the origin must have l or m or both infinite.

Notice the duality implied by this arrangement of point and line co-ordinates. By giving the symbols a different interpretation and taking l, m as co-ordinates of a point, x, y as co-ordinates of a line and bearing in mind the symmetry of the relation of incidence $lx+my+1=0$ in x, y and l, m respectively, we see that to any geometrical theorem corresponds another in which points and lines are interchanged. This is the principle of duality which we have already deduced from the theory of reciprocal polars in Art. 57. The present result shows that this principle is entirely independent of the theory of reciprocal polars.

134. Point and plane co-ordinates in space. In like manner the position of a point P in space may be defined by taking three axes OX, OY, OZ through an origin O and drawing through P planes parallel to YOZ, ZOX, XOY to meet OX, OY, OZ respectively at L, M, N. Then the segments OL, OM, ON taken with proper sign are denoted by x, y, z and called the co-ordinates of the point. It is shown in treatises on analytical geometry (*see* Salmon, *Geometry of Three Dimensions*, or C. Smith, *Solid Geometry*) that in this system of co-ordinates a plane is represented by an equation of the first degree in the co-ordinates which may be put into the form

$$lx+my+nz+1=0 \quad\text{.........................}(1)$$

and conversely that every such equation defines a plane.

(l, m, n) may be called the co-ordinates of the plane and the above equation expresses that the plane (l, m, n) and the point (x, y, z) are incident.

The co-ordinates of a point on a surface satisfy a single equation in x, y, z which is called the Cartesian equation of the surface.

The co-ordinates of a plane tangent to a surface satisfy a single equation in l, m, n which is called the tangential equation of the surface.

The equation (1) expresses, when x, y, z are treated as constants and l, m, n as variables, that the co-ordinates of the planes passing through x, y, z satisfy the equation (1) of the first degree in l, m, n.

Therefore such an equation of the first degree in l, m, n represents a set of planes through a point. Such a set of planes is called a **star of planes** and the point through which they pass is called the vertex of the star.

An equation of the first degree in l, m, n is therefore the tangential equation of a point.

As in Art. 133, $l=0$, $m=0$, $n=0$ are the co-ordinates of the plane at infinity, whereas $x=0$, $y=0$, $z=0$ correspond to infinite plane-co-ordinates.

135. Principle of duality in space. The symmetrical form of the equation

$$lx+my+nz+1=0$$

implies that if the point (x, y, z) and the plane (l, m, n) are incident, so are the plane (x, y, z) and the point (l, m, n). Thus to any theorem connecting points and planes, there corresponds a reciprocal theorem connecting planes and points, obtained from the first by interchanging the interpretations of x, y, z and l, m, n. In this translation the join of two points corresponds to the meet of two planes. Hence a straight line corresponds to a straight line. To the set of lines through a point, which is called a **star of lines,** corresponds the set of lines in a plane, which is called a **plane of lines.** To a star of planes through a point corresponds the set of points of a plane, which is called a **plane of points.** To a range of points on a line corresponds a set of planes through a line or *axis*, which is called an **axial pencil.** To a set of lines through a point and lying in a plane (a flat pencil) corresponds a set of lines lying in a plane and passing through a point (another flat pencil). To a point on a surface corresponds a tangent plane to the corresponding surface. To the tangent plane at a point corresponds the point of contact of the corresponding tangent plane.

To the points where a straight line cuts a surface correspond the tangent planes drawn through a line to the corresponding surface.

136. Cross-ratio of an axial pencil. An axial pencil of four planes α, β, γ, δ through a line x, has a definite cross-ratio. For cut it by any two straight lines u_1, u_2. These meet $\alpha\beta\gamma\delta$ in ranges $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ respectively. On x take two points V_1, V_2. The planes u_1V_1, u_2V_2 meet in a line u_3 which cuts $\alpha\beta\gamma\delta$ in a range $A_3B_3C_3D_3$. Then the ranges $A_1B_1C_1D_1$, $A_3B_3C_3D_3$ are perspective from V_1; and the ranges $A_3B_3C_3D_3$, $A_2B_2C_2D_2$ are perspective from V_2. Hence we have

$$\{A_1B_1C_1D_1\}=\{A_3B_3C_3D_3\}$$

$$\{A_3B_3C_3D_3=\{A_2B_2C_2D_2\}.$$

That is, $$\{A_1B_1C_1D_1\}=\{A_2B_2C_2D_2\}.$$

Hence all straight lines meet an axial pencil of four planes in ranges having the same cross-ratio. This cross-ratio is defined to be the cross-ratio of the axial pencil.

An axial pencil, like a range and a flat pencil, is known as a one-dimensional geometric form of the first order. We shall refer to the axis of such a pencil as its base, and axial pencils with a common axis will be termed cobasal.

137. Imaginary elements. We are now in a position to introduce into Geometry a new set of ideal elements, which are called imaginary elements. In Art. 4 elements at infinity were introduced, in order to enable us to state theorems on the straight line in all their generality, without having to consider cases of exception. Thus, after the introduction of the elements at infinity, we were able to state, quite generally, that coplanar lines always have a point of intersection, that a straight line and a plane always have a point of intersection, that two planes always have a straight line in common.

But, as we proceeded, we met another set of cases of exception which could not be dealt with in the same manner. For example, two collinear projective ranges might have two real self-corresponding points, or they might have none. Nevertheless the nature of two such ranges is not intrinsically different in the two cases, as appears from the fact that any property which we prove for two such ranges which have self-corresponding points holds equally for ranges not having self-corresponding points, provided the property does not involve the reality of the self-corresponding points.

In like manner a straight line may cut a circle or conic in two points, or it may not cut the curve at all. Two tangents may be drawn from a point to a conic, or none may be drawn.

The validity of the results we have reached therefore depends on the elements of the figures having certain relative positions, without which some of the results apparently disappear.

Now it would be extremely convenient if these restrictions could be removed and if, by introducing a new set of ideal elements, which have no visual existence, we could state our theorems in a perfectly general manner.

Such ideal elements are provided for us by the method of co-ordinates explained in Arts. 133, 134.

For any geometrical theorem can be translated into an algebraic theorem connecting point and line co-ordinates (or point and plane co-ordinates). If in this theorem certain real elements appear, the co-ordinates of these elements can be deduced from the solution of certain algebraic equations involving the data. If by altering the numerical values of these data, without altering their nature, these elements disappear from the geometrical theorem, they will not disappear from the algebraic theorem, for an algebraic equation continues to have solutions, even when its constants are such that these solutions are not real. The algebraic solution will therefore still give values for the co-ordinates of those elements which have disappeared from the geometrical solution, but these co-ordinates will be complex, that is of the form $a+ib$, where $i=\sqrt{-1}$ and a, b are real. The points, straight lines or planes defined by such co-ordinates have no visual existence; nevertheless all analytical theorems remain true of them and therefore all geometrical operations, which are interpretable by means of analysis, will continue to hold for such imaginary elements. And this is true not only of points, straight lines and planes, but of all curves and surfaces of higher degree.

Thus the locus

$$x^2+y^2=-a^2$$

is not a real circle: nevertheless it possesses, analytically, all the properties of a circle and, if we admit imaginary elements, we may perform with it the operations which we can perform with an ordinary circle.

We will therefore, from this point onwards, assume the existence of such imaginary elements, so that if a construction which leads to certain elements in one case fails to lead geometrically to such elements in another case, we shall say that those elements are still there, but are imaginary.

Thus we know that two projective collinear ranges will generally have two self-corresponding points. This shows that the problem of determining the self-corresponding points of two such ranges is analytically capable of two solutions. Hence it will have two analytical solutions in *all* cases. We shall then say that two such ranges have *always* two self-corresponding points, but that these may be *real* or *imaginary*.

In the same way a straight line will be conceived as always cutting a conic at two points, real or imaginary; and from a point two tangents, real or imaginary, can always be drawn to a conic.

Again we know that, in general, two distinct conics will intersect in four points. The problem of finding the intersections of two conics has therefore four analytical solutions. We shall say that it has always four geometrical solutions, that is, every two conics have four points of intersection, real or imaginary.

The student may object that this introduction of imaginary elements is really, from the geometrical point of view, a mere verbal delusion, for in what way can we derive help in practice from a construction in which one or more steps are imaginary? The answer is that these imaginary elements cannot, indeed, be used in drawing-board *constructions*, but it may, and does, happen that a *demonstration*, involving such imaginary elements, leads to a result which is free from them. Thus by means of imaginary points and lines we can obtain real theorems, precisely as we can, by means of points and lines at infinity, obtain theorems relating to figures at a finite distance.

It is true that in all cases such theorems might be obtained by reasoning with purely real elements. But such proofs are often exceedingly complicated; also two theorems which, when we use imaginary elements, are only particular cases of the same theorem, require, if we restrict ourselves to real elements, proofs which are not infrequently quite dissimilar. The simplicity and unity obtained by the introduction of imaginary elements add very greatly in power to the methods of geometry.

138. Conjugate imaginaries. If the co-ordinates of an element are of the form $a+ib$, the element whose co-ordinates are obtained from those of the first by changing the sign of i is said to be a conjugate imaginary to the first element.

Thus the point $(0, \; -i, \; 1+i)$ is the conjugate imaginary point to $(0, i, 1-i)$.

If two elements are incident, their conjugate imaginary elements are also incident.

For any equation involving imaginaries may be reduced to the form $U+iV=0$, where U and V are real. We have therefore $U=0$, $V=0$, and therefore $U-iV=0$, that is, the equation obtained by changing the sign of i everywhere is also satisfied.

It follows similarly that if a real and an imaginary element are incident, the real element and the conjugate imaginary element are also incident. For a real element may be looked upon as its own conjugate imaginary.

If an element A of any nature is determined by two other elements P, Q (points, planes or intersecting lines), its conjugate imaginary element A' is determined by the conjugate imaginary elements P', Q'. For since A, P are incident $\therefore A'$, P' are incident; and since A, Q are incident $\therefore A'$, Q' are incident. Hence $A'=P'Q'$. In particular if $Q=P'$ $\therefore Q'=P$ or $A'=P'P=A$. Hence the element (if any) determined by two conjugate imaginary elements is always real.

In particular the join of two conjugate points or the meet of two conjugate planes is a real line. Two conjugate lines which intersect determine a real point of intersection and a real plane.

The elements determined by a real element A and two conjugate imaginary elements P, P' are conjugate imaginary. For A being its own conjugate imaginary, AP is the conjugate imaginary to AP'.

Also, if S be any locus or envelope which is real or into whose analytical equation only real coefficients enter, and P be any imaginary element incident with S (*i.e.* lying on or tangent to S), the relation of incidence is expressed by an equation

$$U + iV = 0.$$

This implies $$U - iV = 0.$$

But the latter is what we obtain if we change the sign of i in the co-ordinates of P, since the coefficients of the equation for S do not contain i. Hence P' is also incident with S.

It follows that if two real loci have one imaginary intersection P, the conjugate imaginary point P' is also an intersection, since it must lie on both curves. The corresponding chord PP', being determined by two conjugate elements, is real.

EXAMPLES

1. Prove that, if I, J are two conjugate imaginary points on a straight line, the middle point O of IJ is real and OJ^2 is real and negative.

2. If I, J are two conjugate imaginary points on a straight line, prove that they can be obtained as the imaginary double points of an elliptic involution.

3. Show that any two conjugate imaginary elements of a form with a real base can be obtained as the double elements of an elliptic involution on that base.

4. Prove that if an imaginary line l do not intersect its conjugate imaginary l', the line drawn from a real point P to meet l and l' is always real.

5. Show that the reciprocal elements of two conjugate imaginary elements are themselves conjugate imaginary when the reciprocal elements of real elements are real.

6. Show that conjugate imaginary elements project into conjugate imaginary elements when the projection is real.

139. Number of real elements incident with an imaginary element. An imaginary point has only one real line through it, namely the one joining it to its conjugate imaginary point. For if it had two it would be the intersection of two real lines and therefore a real point.

Similarly an imaginary plane has only one real line lying in it, namely its intersection with its conjugate imaginary plane. For a plane through two real lines is a real plane.

An imaginary line, for a similar reason, cannot have two real points on it. But imaginary lines may be of two kinds. A line of the first kind has one real point on it. A line of the second kind has no real point on it.

By the last Article the conjugate imaginary line p' to a line p of the first kind passes through the real point on p. p, p' therefore intersect and, being conjugate, determine a real plane.

Thus a line of the first kind has one real plane passing through it. It cannot have a second, for it would then be the meet of two real planes and so be a real line.

Conversely, if an imaginary line p has one real plane passing through it, its conjugate imaginary line p' lies in this plane and meets p at a real point, so that p is of the first kind.

A line of the second kind has therefore no real plane through it, as well as no real point on it, and it does not intersect its conjugate imaginary line.

Such lines may be obtained by taking conjugate imaginary pairs P, P' and Q, Q' on non-intersecting real lines a, b respectively. Then P, P', Q, Q' cannot be coplanar and the lines PQ, $P'Q'$ are conjugate imaginary lines which do not intersect.

140. The circular points at infinity. Consider the two (imaginary) points in which the line at infinity i^∞ in a plane meets any circle in the plane. Since the pole of i^∞ is the centre C of the circle the involution of conjugate points on i^∞ is given by the intersection of i^∞ with the involution of conjugate rays through C. But since conjugate diameters of a circle are at right angles (Art. 54) the latter involution is the rectangular involution through C.

The two intersections Ω, Ω' of i^∞ with the circle are therefore the double points of the involution in which the rectangular involution through C meets i^∞.

But if we take any other point O in the plane and join O to the points of the involution on i^∞ we obtain an involution through O whose rays are parallel to the corresponding rays of the involution through C. The involution through O is therefore also rectangular. Thus the double rays of all rectangular involutions pass through the same two points Ω, Ω' at infinity. These points Ω, Ω' are therefore determined quite independently of the particular circle chosen. Hence all circles pass through the same two points Ω, Ω'.

Conversely every conic which passes through Ω, Ω' is a circle. For let s be such a conic and let A, B, C be any three other points on s. Describe the circle c through A, B, C. Then it passes through Ω, Ω'. c and s have five points A, B, C, Ω, Ω' common and therefore coincide.

For these reasons the points Ω, Ω' are called the **circular points at infinity.** Being the intersections of a real line (the line at infinity) with a real curve, they are conjugate imaginary points by Art. 138.

Two interesting cases of circles arise when the conic through Ω, Ω' degenerates into a line-pair. If Ω, Ω' are on the same component of the pair, the latter consists of the line at infinity and an accessible straight line. Thus any straight line, together with the line at infinity, may be regarded as forming a circle of infinite radius.

If Ω, Ω' be on different components of the pair we see that any pair of lines through Ω, Ω' form a circle.

If their point of intersection P be real, every line through P is a tangent to the curve at P (*see* Art. 44) and the circle is then a point-circle (*see* Art. 112).

141. Circular lines. The lines joining any point of the plane to Ω, Ω' are called the **circular lines** through the point. If the point be real, the circular lines through it are conjugate imaginaries. From the last Article the circular lines through a point are the double rays of the rectangular involution through the point.

Hence any pair of lines at right angles are harmonically conjugate with regard to the circular lines through their intersection.

It follows that if in any involution pencil the circular rays are mates, the double rays are at right angles. Conversely if the double rays are at right angles the circular rays are mates.

Since by Art. 52 the self-corresponding rays of conjugate pencils through a point C, that is, the double rays of the involution of conjugate lines through C, are the tangents from C to the conic, it follows that, when the conic is a circle and C its centre, so that the conjugate lines through C form a rectangular involution, the circular lines through C are the tangents from C to the circle, and the points Ω, Ω', where they meet the polar of C, are their points of contact. Thus **two concentric circles have the same tangents at Ω, Ω' and are to be regarded as touching one another at these points.**

142. The arms of an angle of given magnitude determine with the circular lines through its vertex a constant cross-ratio. Consider an angle of given magnitude rotating about its vertex O. Its arms trace out two directly equal concentric flat pencils of which the self-corresponding rays are by Art. 91 parallel to the asymptotes of a circle, that is, they are the circular lines through O. Thus if POP', QOQ' be any two positions of the angle, (OP, OP'), (OQ, OQ') are two pairs of corresponding rays; they determine therefore the same cross-ratio with the circular lines through O (Art. 88).

If on the other hand two angles with different vertices O, O' have their arms parallel, the parallel arms and the circular lines through O, O' determine the same range on the line at infinity. They form two perspective flat pencils and the cross-ratios are the same.

Combining the above two results, if an angle of given magnitude be moved about in its own plane anyhow, it defines a fixed cross-ratio with the circular lines through its vertex.

The converse theorem that, if a moving angle determine with the circular lines through its vertex a constant cross-ratio, the magnitude of the angle is fixed, is readily proved.

Examples

1. Show that any two concentric projective pencils in a plane can always be projected into directly equal pencils.

2. Show, by considering the circle as the product of two directly equal pencils and applying the construction of Art. 91 for its asymptotes, that each of the circular lines through a point may be looked upon as making any given angle with itself.

3. Show analytically that the circular lines are parallel to the lines $y = \pm ix$ and verify that they make the same angles $\pm \tan^{-1} i$ with every straight line in the plane.

4. Discuss the form assumed by the anharmonic property of four fixed points and one variable point on a conic, when two of the fixed points are the circular points.

5. Prove that if OA, OB be two lines intersecting at O the cross-ratio $O\{A\Omega B\Omega'\} = e^{2i\theta}$, where $\theta =$ angle AOB.

143. The circular points are conjugate with regard to any rectangular hyperbola. For in a rectangular hyperbola the double rays of the involution of conjugate diameters are at right angles. Therefore the circular lines through the centre are conjugate. The points where they meet the polar of the centre (*i.e.* the line at infinity) are therefore also conjugate with regard to the hyperbola. But these are the circular points Ω, Ω'.

Conversely if Ω, Ω' are conjugate points with regard to a hyperbola, the circular lines through the centre are conjugate lines and the asymptotes are at right angles.

144. The orthoptic circle. Consider the pencils of conjugate rays with respect to any conic s through the circular points Ω, Ω'. These pencils are projective by Art. 52. Their product is therefore a conic passing through Ω, Ω', that is, a circle.

Let P be any point on this circle. Then $P\Omega$, $P\Omega'$ being lines through P conjugate with regard to s are harmonically conjugate with regard to the two tangents from P to s (Art. 52). Therefore these two tangents are at right angles (Art. 141).

Conversely if these two tangents are at right angles $P\Omega$, $P\Omega'$ are mates in the involution of conjugate rays through P, and P lies on the product of the conjugate pencils through Ω, Ω'. We have then the theorem :

The locus of the intersection of perpendicular tangents to a conic is a circle.

The circle is called the **orthoptic circle** of the conic, from the property that at any point of it the conic subtends a right angle. It is also called the **director circle,** by analogy with its degenerate case when the conic is a parabola, when the locus of intersections of tangents at right angles is the directrix (Art. 128). The explanation of this from our point of view is that in the case of the parabola $\Omega\Omega'$ touches the curve and is therefore a self-corresponding ray of the conjugate pencils through Ω, Ω'. These are accordingly perspective and the locus breaks up into $\Omega\Omega'$ (the line at infinity) and another straight line, which is the directrix.

The orthoptic circle is concentric with the conic. For the tangent at Ω to the orthoptic circle is the line through Ω conjugate to $\Omega\Omega'$ with regard to the conic (Art. 39). It must therefore pass through the pole of $\Omega\Omega'$, *i.e.* through the centre of the conic. Similarly the tangent at Ω' to the orthoptic circle passes through the centre of the conic. The pole of $\Omega\Omega'$ with regard to the circle (*i.e.* the centre of the circle, $\Omega\Omega'$ being the line at infinity) is thus the centre of the conic.

The radius of the orthoptic circle is immediately found by drawing the (perpendicular) tangents at the extremities of the axes. The semi-diagonal of the rectangle so formed is the radius required. It is $\sqrt{CA^2+CB^2}$. In the hyperbola $CB^2=-CB_1{}^2$, so the radius of the orthoptic circle $=\sqrt{CA^2-CB_1{}^2}$. If $CB_1{}^2>CA^2$

the orthoptic circle is imaginary. If $CB_1^2 \doteq CA^2$, or the hyperbola is rectangular, it shrinks into a point at the centre. Thus the only real perpendicular tangents to a rectangular hyperbola are the asymptotes.

145. The four foci of a conic. By the definition of a focus the involution of conjugate lines through it is rectangular. Thus the tangents from a focus to the conic, being the double rays of such an involution, are the circular lines through the focus and pass through Ω, Ω'. Conversely a point F which is the intersection of tangents from Ω, Ω' must be a focus, for the double rays of the involution of conjugate rays through F will be the tangents from F, namely $F\Omega$, $F\Omega'$. But these being the circular lines, the involution defined by them must be rectangular, or F is a focus.

Since two tangents t_1, t_2 can be drawn to a conic from Ω and two tangents t_1', t_2' can be drawn from Ω', a conic will have *four* foci, namely $t_1t'_1$, t_1t_2', t_2t_1', t_2t_2'. Of these two are real and two imaginary, as follows. Take one tangent t_1 from Ω. This being an imaginary line in a real plane, has a real point F_1 on it (Art. 139). The other tangent from F_1 must be a conjugate imaginary line to t_1, for two imaginary tangents from a real point to a real conic must be conjugate imaginaries, as can be shown from reasoning similar to that used in Art. 138 to prove that intersections of a real line and a real conic are conjugate imaginaries.

This other tangent from F_1, being a conjugate imaginary to t_1, *i.e.* to $F_1\Omega$, must be $F_1\Omega'$. Call it then t_1'. Let t_2 be the other tangent from Ω. If F_2 be the real point on it, then $F_2\Omega' = t_2'$, and t_2, t_2' are conjugate imaginary lines. F_1, F_2 are the two real foci of the curve. t_1t_2', t_2t_1', which we may call F_3 and F_4, are the intersections of non-conjugate imaginary lines and are imaginary points. They are, however, themselves conjugate imaginary points, being intersections of two conjugate imaginary pairs (Art. 138). Hence F_3F_4 is a real line.

Now by Art. 50 the diagonal triangle of the complete quadrilateral $t_1t_1't_2t_2'$ circumscribed to the conic is self-polar with regard to the conic. But the sides of this diagonal triangle are F_1F_2, F_3F_4, $\Omega\Omega'$. The meet of F_1F_2, F_3F_4 is therefore the pole of $\Omega\Omega'$, *i.e.* the centre C of the conic; F_3F_4, F_1F_2 are then conjugate diameters. By the harmonic property of the complete quadrangle $F_1F_3F_2F_4$ the two sides of the diagonal triangle through C, viz. $C\Omega$, $C\Omega'$, are harmonically conjugate to the two sides of the quadrangle through C, namely F_1F_2, F_3F_4. $C\Omega$, $C\Omega'$ being circular lines F_1F_2, F_3F_4

are perpendicular and so must be axes. The two imaginary foci therefore lie on what we have called hitherto the non-focal axis of the curve.

They are, in fact, the imaginary intersections of this axis with the conic k of Art. 115, and the double points of the elliptic involution of orthogonal points on the non-focal axis (Art. 116). Any two rectangular conjugate lines meet this axis in a real pair of mates Q, Q' of this involution, of which C is the centre, and F_3, F_4 the double points. We have therefore

$$CF_3^2 = CQ.CQ' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Thus CF_3^2 is real, and so must be negative, or F_3 would be real. Clearly CF_3 is the semi-diameter of the rectangular hyperbola k perpendicular to CF_1 (F_1 being a real focus). By Art. 83, CF_1 and CF_3 must be equal in absolute length.

Thus $$CF_3^2 = -CF_1^2 = -e^2.CA^2 \quad \ldots\ldots\ldots\ldots\ldots(2)$$

146. The two eccentricities of a conic. The reasoning of Art. 118, which establishes the eccentricity property, still holds good formally of the foci F_3 and F_4 and of the corresponding (imaginary) directrices; an eccentricity e' corresponding to these foci therefore exists, but now the discrimination of the different types of conic from the reality of the intersections of the circle with the vanishing line has no longer any meaning.

We may find this eccentricity e' as follows. If the non-focal axis, on which F_3 lies, meets the curve at B, B' and the directrix corresponding to F_3 at Y, we have, since $\{F_3BYB'\} = -1$,

$$CF_3.CY = CB^2$$

Now $e' = F_3B : BY = (CB - CF_3) : (CY - CB)$

$$= (CB - CF_3) : \left(\frac{CB^2}{CF_3} - CB\right) = CF_3 : CB,$$

so that $CF_3 = e'.CB$, which gives a relation symmetrical with $CF_1 = e.CA$. Using (2) of Art. 145, we have

$$e'^2.CB^2 + e^2.CA^2 = 0,$$

or $$e'^2(1 - e^2) + e^2 = 0,$$

which leads to the more symmetrical form

$$\frac{1}{e^2} + \frac{1}{e'^2} = 1$$

A conic has therefore two eccentricities, corresponding to the two pairs of foci, and connected by the above relation. If the conic is an ellipse, e' is a pure imaginary; for the hyperbola e and e' are both real and greater than unity. For the parabola $e' = \infty$.

147. Confocal conics. If two foci F_1, F_2 of a conic lying on the same axis be given, the other foci F_3, F_4 are determined. For they are the remaining vertices of the complete quadrilateral formed by the four lines $F_1\Omega$, $F_1\Omega'$, $F_2\Omega$, $F_2\Omega'$.

In particular, conics which have the same two real foci have all their foci the same. Such conics are called **confocal conics.** They touch four fixed lines, namely the sides of the quadrilateral mentioned above.

148. The circular points are foci of a parabola. In the case of a parabola the line at infinity $\Omega\Omega'$ is a tangent. Thus t_2, t_2' coincide with $\Omega\Omega'$. The quadrilateral of tangents from Ω, Ω' reduces therefore to a triangle. F_1, *i.e.* (t_1t_1'), remains as the only accessible real focus of the curve, F_2 is the point of contact of the line at infinity, *i.e.* the point at infinity on the axis. F_3 and F_4 become intersections of t_1 and t_1' with the line at infinity, that is, they coincide with Ω, Ω' which are thus foci of the curve.

We have therefore an exception to the theorem of the last Article, for the giving of Ω, Ω' does not here determine the other foci.

EXAMPLE

Show that two parabolas with a common focus, but different axes, have one real accessible common tangent, and one only.

Discuss the case where both the focus and axis are common.

149. Imaginary projections. By means of the circular points a number of important theoretical results in projection can be deduced.

Thus any two conics can always be projected simultaneously into circles.

For let A, B be any two of the intersections of such conics. Then by projecting A, B into the circular points in any plane, the conics are projected into circles.

This result is of great importance, since it enables us to apply to a pair of conics any projective theorem proved for a pair of circles.

This projection of two given points into the circular points is of course imaginary if the two given points are real. If the two given points are conjugate imaginary points, they will in general be given as the intersections of a real straight line x with a real conic s, when x and s do not cut in real points. Take O the pole of x with regard to s and two pairs (OP, OP'), (OQ, OQ') of conjugate lines through O with regard to s. Project x to infinity and the angles POP', QOQ' into right angles (Art. 19). O projects into the centre of the conic and (OP, OP'), (OQ, OQ') into pairs of conjugate diameters at right angles, *i.e.* into axes. But since a conic with more than one pair of axes must be a circle, s projects into a circle and its intersections with x into the intersections of a circle with the line at infinity, that is, into the circular points. Thus a real projection transforms a pair of conjugate imaginary points into the circular points.

Alternatively the two conjugate imaginary points may be given as the double points of an elliptic involution on a real line x. Take two pairs of mates (P, P'), (Q, Q') of this involution, and join to any point O outside the line. Then proceed as before.

Again two conics can always be projected simultaneously into rectangular hyperbolas, by projecting two of their common points, A, B into rectangular points at infinity. This can be done in an infinite number of ways by taking AB as vanishing line in a plane perspective, and the pole of perspective to be any point of the circle on AB as diameter. If A, B are real, the perspective is real.

Also two conics can always be projected into two confocal conics, by taking two opposite vertices of the complete quadrilateral formed by their common tangents and projecting these vertices into the circular points. The two projected conics have the same tangents from the circular points and are therefore confocal. Accordingly all projective properties of confocal conics are properties of any pair of conics.

An important case of frequent occurrence is that of two conics touching one another at two points A and B. By projecting A, B into Ω, Ω' the conics transform into circles. The pole of AB, which is clearly common to the two conics, transforms into the pole of the line at infinity $\Omega\Omega'$, so that the circles are concentric, and any projective proposition which is true of concentric circles is also true of any two conics having double contact, real or imaginary. The conics, of course, may not have real common points at all.

150. The eight tangents to two conics at their four common points touch a conic. We will take an example of the deduction of theorems for two conics from theorems for two circles.

Let two circles whose centres are S, O (Fig. 53) intersect at A and B. A and B are symmetrically situated with regard to SO. Let C be the middle point of SO. The circle whose centre is C and which passes through A also passes through B. Construct the conic having S, O for foci and the circle with centre C and radius CA for auxiliary circle. This conic touches the tangent at A to the circle centre S, for this tangent is perpendicular to SA and A is a point on the auxiliary circle of the conic (*see* Art. 125). Similarly the conic touches the tangent at B to the circle centre S and the tangents at A, B to the circle centre O.

Consider now the other intersections of the two given circles,

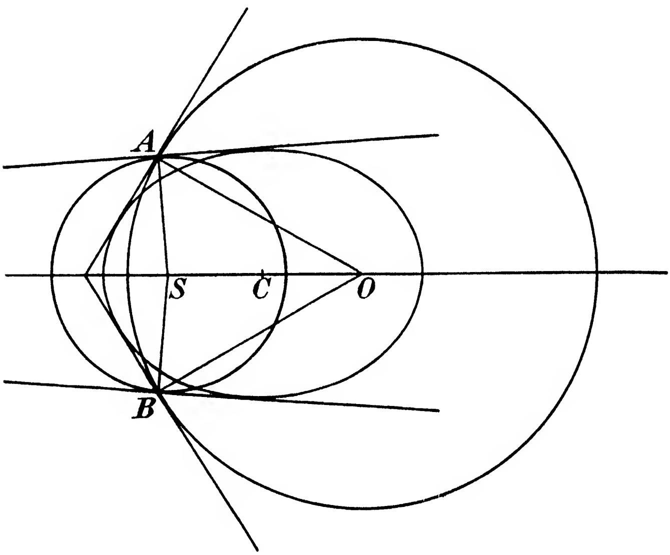


Fig. 53.

namely Ω, Ω'. The tangents to the circle centre S at Ω, Ω' pass through S since S is the pole of $\Omega\Omega'$ with regard to the circle. They are therefore $S\Omega$, $S\Omega'$. But these are also tangents to the conic, since S is a focus.

In like manner the tangents at Ω, Ω' to the circle centre O are tangents to the conic.

Hence the eight tangents at the four common points of two circles touch a conic. Projecting the circles back into any two conics we obtain the result :

The eight tangents to two conics at their four common points touch a conic.

Reciprocating this theorem we obtain the following :

The eight points of contact of the four common tangents to two conics lie on a conic.

EXAMPLES IXA

1. Prove that, if the circular lines through a point P on a conic meet the conic again at Q, Q', then QQ' is the polar of the Frégier point of P and is equally inclined to the axes with the tangent at P.

2. If P_1OQ_1, P_2OQ_2 are two conjugate diameters of a conic, show that it is possible to give a rule enabling us to select an extremity of each, so that the ranges $[P_1]^2$, $[P_2]^2$ are projective.

[Project the points at infinity on the asymptotes into the circular points.]

3. Show that four conics can be drawn through two fixed points I, J to touch the sides of a given triangle ABC.

If P, Q, R, S are the poles of IJ with regard to these four conics, show that the triangle ABC is self-polar with regard to any conic through P, Q, R, S.

4. Two conics k_1, k_2 have double contact and a tangent t to k_1 meets k_2 at P, P'. Show that $[P]^2 \barwedge [P']^2$.

5. Two conics k_1, k_2 have double contact and from a point P of k_1 tangents t, t' are drawn to k_2. Show that $[t]^2 \barwedge [t']^2$.

6. A and B are two fixed points on a conic and PT, PT' the tangents from a variable point P. Prove that if the cross-ratio of the pencil $P(ABTT')$ is constant the locus of P consists of two other conics touching the given conic at A and B.

7. Show that if P and Q be the two distinct points of contact of a common tangent to two conics which touch at L, and R be the point at which the tangent at L to the conics meets PQ, then $\{RPUQ\} = -1$, U being the point where the chord through the other intersections of the two conics meets the common tangent.

[Project the two conics into circles.]

8. Prove that the polars, with respect to the conics which touch CA, CB at A, B respectively, of a given point Q in the plane of the triangle ABC concur at a point of the line AB.

If P is the point of contact of a tangent from Q to one of these conics, prove that PQ, PC are harmonically conjugate with respect to PA, PB ; and show that the locus of P is the conic through Q, A, B, C, for which QC is conjugate to AB.

Express this theorem in metrical form when the conics are rectangular hyperbolas with the same asymptotes.

[For first two parts project A, B into Ω, Ω'.]

9. Prove from Art. 138, Ex. 2 that any two conjugate imaginary points in a plane can be projected into the circular points by a real projection.

10. Prove that, if two real conics have only two real intersections, they have only two real common tangents and conversely.

11. Show that the four points where the tangents from Ω, Ω' touch a conic lie on the orthoptic circle.

12. Prove that any point P on a conic and the pole of the normal at P are conjugate points with regard to the orthoptic circle of the conic.

13. Prove that a conic is uniquely determined when its orthoptic circle and two tangents, not at right angles, are given. Show how to construct the foci from the above data.

14. A, B, C are three points on a conic s. Show that the lines through A, B, C which are conjugate respectively to BC, CA, AB with regard to s meet at a point H.

The point A is fixed on s, and B and C are a variable pair of a given involution on s, whose double points are L and M. Show that the locus of H is a conic s', which passes through L and M and touches s at A.

Prove also that if now the point A moves on s, then the conic s' moves in contact with s and with a fixed conic which touches s at L and M.

[Project L, M into the circular points.]

15. P, Q are any two real points inside a real conic s, and (p, p'), (q, q') are the imaginary tangents from P, Q to s. Prove that the diagonal triangle of the complete quadrilateral $pp'qq'$ is entirely real, and show how to construct it.

Hence prove that, by a real projection s and P, Q may be transformed into a real conic s_1 and two real points P_1, Q_1 which are the real foci of s_1.

16. A conic passes through three real points A, B, C and the imaginary double points of a given elliptic involution on a real straight line x. Show how to construct at least two other real points on the conic (and therefore any number of such points).

17. A conic touches three real lines a, b, c and the imaginary double rays of a given elliptic involution pencil of vertex O. Show how to construct the conic by tangents.

18. A conic passes through a real point A and the imaginary double points of two given elliptic involutions on real lines u, v intersecting at O. Find the two tangents from O and their points of contact, and hence construct the conic.

[Conjugate ranges on the common mates p, q of the involution pencils through A incident with the given involutions have A for a self-corresponding point and are perspective from O. If X, Y are the mates of O in the given involutions, XY is the polar of O and meets p, q at the points of contact P, Q of tangents from O to the conic, since on OP there are at least two points, namely O and (q, OP) which are conjugate to P; and similarly for Q.]

19. A conic touches a real line a and the imaginary double lines of two given elliptic involution pencils of vertices U, V. Find the intersections of UV with the conic and the tangents at these intersections.

Interpret your construction when the involution pencils are rectangular, and state the theorem to which it leads.

20. A, B, X, Y are four fixed points on a conic. Show that a point O and a straight line c can be found in the plane of the conic such that, if U is a variable point of the conic and UA, UB meet c at P, Q respectively, then $O\{PQXY\}$ is equal to a given cross-ratio.

What does the above theorem become when X, Y are the circular points ?

21. Show how to determine the conic which passes through a given point A and touches a given conic k at its imaginary intersections with a given real line l.

22. If any tangent to a conic, whose centre is C, meets the orthoptic circle at P and Q, show that CP, CQ lie along conjugate diameters.

23. Prove that the joins of pairs of corresponding points of two projective

ranges on the same conic k envelop a second conic, having double contact with k.

The sides AB, BC, CD of a simple quadrilateral $ABCD$ inscribed in a given conic pass respectively through fixed points J, K, L. Prove that, in general, DA touches a second fixed conic; but that, if J, K, L are collinear, then DA passes through a fourth fixed point in the line JKL.

24. a and b are two tangents to a conic s, x any line through their intersection, P and Q a pair of points on x, conjugate with respect to s. A conic s' is drawn to pass through P and Q and the points of contact of the tangents a and b. Prove that the tangents to s' at the other pair of common points of s and s' intersect at ab.

25. The tangents to a conic k at fixed points A, B meet at C and P, Q are any two points of k. Prove that $P(ABCQ) \barwedge Q(ABPC)$; and show that, if P, Q move on k so that $P\{ABCQ\} = \text{const.}$, the chord PQ is always tangent to a certain fixed conic, which touches k at A and B.

26. If (p, p') are a pair of mates in the involution of lines through a fixed point O conjugate for a given conic k, and if p, p' meet a tangent t to the conic at P, P', then if P describes a fixed line l, P' describes a conic having double contact with k.

[Project into the circular points the points of intersection of k with the line through O conjugate to l.]

EXAMPLES IXb

[The axes of co-ordinates are rectangular throughout.]

1. A conic has $x^2+y^2=9$ for its orthoptic circle. If the conic touches the line $x+2y=3$ at the point $(1, 1)$, construct its foci and axes.

2. A parabola touches the x-axis at the origin O, touches the line $x=1$, and has its axis parallel to the line $x+2y=0$. Find the point of contact of the line $x+2y=2$ with the conic c which touches this line and has four-point contact with the parabola at O. Construct the orthoptic circle of c.

3. A conic passes through the points $(4, 5)$, $(0, 1)$, $(2, 2)$ and the pairs of points $(0, 0)$, $(4, 0)$; $(-1, 0)$, $(2, 0)$ are conjugate with regard to it. Construct two more points on the conic and the tangents at these points.

4. A conic passes through the point $(2, 3)$; the following pairs of points are conjugate with regard to it: $(-2, 0)$, $(1, 0)$; $(0, 0)$, $(3, 0)$; $(0, 0)$ $(0, 2)$; $(0, 1)$ $(0, 5)$.

Construct the tangents to this conic from the origin and their points of contact.

5. A conic passes through the point $(0, -3)$ and has double contact with the circle $(x-1)^2+(y-5)^2=3^2$ at the imaginary points where it is met by the axis of x. Obtain four other real points on the conic.

6. The semi-axes of an ellipse are 4 cm. and 3 cm. respectively. With an extremity of the major axis as centre and radius 5 cm. a circle is described. Construct the two real common chords of the circle and ellipse.

CHAPTER X

HOMOGRAPHY

151. Homographic ranges. Let x be the distance of a point P on a line u from a given origin O on the line. Let x' be the distance of a point P' on another line u' from an origin O' on that line.

Let a correspondence be established between the ranges of such a nature that to any point P (real or imaginary) corresponds one point P' (real or imaginary) and one only, and conversely to every point P' corresponds one point P and one only. And let the correspondence be algebraic, that is, let the relation between P and P' be expressible by means of a rational integral algebraic equation between x and x', that is, an equation in which only sums of positive powers or of products of positive powers of x and x' appear equated to zero. No transcendental functions such as $\sin x$, $\log x$, e^x, etc., are to appear in the relation between x and x'.

Since for a given value of x there is one value of x' and one only, the equation can involve only the first power of x'; and since for a given value of x' there is only one value of x, it can involve only the first power of x.

It will therefore take the form

$$Axx' + Bx + Cx' + D = 0 \quad \text{.........................(1)}$$

Two ranges between which such a one-one correspondence exists are said to be **homographic.**

Projective ranges are clearly homographic: for their correspondence is one-one and the relation between the co-ordinates of a point and of its projection on any plane is certainly algebraic and rational.

152. Homographic ranges are equi-anharmonic. The relation (1) of Art. 151 leads to

$$x' = -\frac{Bx + D}{Ax + C}.$$

Let x_1, x_2, x_3, x_4 be the x's of four points P_1, P_2, P_3, P_4 on u; and x_1', x_2', x_3', x_4' the x''s of the four corresponding points P_1', P_2', P_3', P_4'.

Then $$\{P_1'P_2'P_3'P_4'\} = \frac{P_1'P_2'.P_3'P_4'}{P_1'P_4'.P_3'P_2'} = \frac{(x_2'+x_1')(x_4'-x_3')}{(x_4'-x_1')(x_2'-x_3')}.$$

Now $$x_2'-x_1' = -\frac{Bx_2+D}{Ax_2+C}+\frac{Bx_1+D}{Ax_1+C}$$
$$= \frac{(AD-BC)(x_2-x_1)}{(Ax_2+C)(Ax_1+C)}.$$

Hence $$\frac{(x_2'-x_1')(x_4'-x_3')}{(x_4'-x_1')(x_2'-x_3')} = \frac{(x_2-x_1)(x_4-x_3)}{(x_4-x_1)(x_2-x_3)},$$
the other factors all cancelling. Therefore
$$\{P_1'P_2'P_3'P_4'\} = \{P_1P_2P_3P_4\},$$
or in homographic ranges corresponding sets of four points have the same cross-ratio.

It follows from the above that homographic ranges are projective. For given two homographic ranges construct two projective ranges having two corresponding triads the same as in the two homographic ranges. Then since both the projective and the homographic relation are equi-anharmonic, to any fourth point of one range will correspond the same fourth point of the other, whether projectively or homographically. The two given homographic ranges are therefore projective ranges.

It follows that, if in two homographic ranges, the elements of one pair correspond doubly, the same is true of all pairs of such elements (*see* Art. 95) and the two homographic ranges form an involution.

In this case equation (1) of Art. 151 must be symmetrical in x, x', that is, of the form
$$Axx' + B(x+x') + C = 0.$$
If x, x' are the roots of the quadratic
$$\alpha\xi^2 + \beta\xi + \gamma = 0,$$
then $x+x' = -\beta/\alpha$, $xx' = \gamma/\alpha$, so that the necessary and sufficient condition that the above quadratic should determine a pair of mates in the given involution is that α, β, γ satisfy the linear equation
$$A\gamma - B\beta + C\alpha = 0.$$

Thus the quadratics whose coefficients satisfy a linear equation define the pairs of mates in an involution.

153. Homographic flat pencils. If the rays of two flat pencils are connected by a one-one correspondence such that, if m be any parameter in terms of which the co-ordinates of any ray of one pencil can be expressed linearly (and, conversely, which is uniquely determined when this ray is given), and if m' be a similar parameter for the corresponding ray of the other pencil, then m and m' are related by a rational algebraic equation linear in both parameters and therefore of the form

$$Amm' + Bm + Cm' + D = 0,$$

the two flat pencils are said to be homographic.

Usually m, m' are the tangents of the angles made by the rays with fixed lines in the planes of the pencils.

It is clear that the ranges in which two such pencils will cut any transversals u, u' are likewise homographic. For the distances x, x' of the points of section measured along u, u' are connected with m, m' (and therefore with each other) by rational algebraic relations; and also the correspondence between x, x' is seen to be one-one.

Since these homographic ranges are equi-anharmonic and projective, the two homographic pencils which stand on these ranges are also equi-anharmonic and projective.

Conversely projective pencils are homographic, since the correspondence between the rays is one-one and the relation between the co-ordinates of corresponding rays must clearly be both algebraic and rational.

Here again, if the relation is symmetrical in m, m', and therefore of the form

$$Amm' + B(m + m') + C = 0,$$

then every pair of elements correspond doubly, and the two homographic pencils form an involution pencil.

So far the term "homographic" has been found to be synonymous with "projective." We now pass on to cases where it enlarges the notion of projective forms.

154. Homographic axial pencils. In like manner two axial pencils whose planes correspond uniquely, while the co-ordinates of corresponding planes are connected by an algebraic relation, are said to be homographic.

The flat pencils in which two homographic axial pencils meet any two given planes are themselves homographic and projective.

The ranges in which two homographic axial pencils are met by any two given straight lines are homographic and projective.

Note that we cannot use the term projective of homographic axial pencils, since these are not plane forms and cannot therefore be projected into one another.

Two homographic axial pencils are entirely determined by two corresponding triads. For take two straight lines meeting the axial pencils in projective ranges, two corresponding triads of planes of the axial pencils determine on the lines two corresponding triads of points of the ranges. These determine the relation between the ranges and therefore the relation between the axial pencils.

Notice that if two homographic axial pencils have a common axis they have two self-corresponding planes, which correspond to the two self-corresponding points of the projective ranges in which the axial pencils are cut by any straight line.

155. Involution axial pencil. We may now apply our definition of involution (Art. 95) to an axial pencil. If we have two cobasal axial pencils (the base, in this case, being the axis) which are homographic, and such that one pair of corresponding planes correspond doubly, then all pairs of corresponding planes correspond doubly, and their aggregate constitutes an involution axial pencil.

The properties of an involution of planes through an axis are closely similar to those of an involution of coplanar rays through a point. Such an involution determines corresponding involutions, of points on any straight line which cuts it, of rays on any plane which cuts it. By constructing the double elements of either of these the double planes of the axial pencil may be found. As before, if two involutions of planes have the same axis they have one pair of common mates, which is always real unless the two given involutions have two pairs of real double planes which are separated by one another.

The relation between the dihedral angles of six planes in involution is found by taking a section by a plane perpendicular to the axis. The angles of the flat pencil so found measure the dihedral angles of the axial pencil. These are therefore connected by the formulæ of Art. 102.

Also planes at right angles form an involution of which the double planes pass through the circular points at infinity in the plane perpendicular to the axis.

Precisely as is done in Art. 109 we can show that every involution of planes through an axis has one pair of perpendicular elements.

156. Homographic unlike forms. If there be a one-one algebraic correspondence between the rays of a flat pencil and the points of a range, the two forms will still be spoken of as homographic.

Similarly a range and an axial pencil, or an axial pencil and a flat pencil, may be homographic.

Clearly from two unlike homographic forms may be derived, by projection or section, two like homographic forms.

A particular case of homographic unlike forms is furnished by the principle of duality, the correspondence between any element and its reciprocal being obviously one-one and algebraic.

Also what we have called incident forms are necessarily homographic. Thus a flat pencil is homographic with the range which it determines on any straight line.

157. Homographic ranges and pencils of the second order. If there be a one-one correspondence between the elements of two forms of the second order (ranges or pencils) which is expressible by an algebraic relation between the co-ordinates of the elements the forms are said to be homographic. A form of the second order may also be homographic with a form of the first order.

It is easy to show that if the forms of the second order are both ranges, or both pencils, such homographic forms are projective forms of the second order as defined in Art. 85.

For example, if we join two homographic ranges $[P_1]^2$, $[P_2]^2$ to vertices O, S lying on their respective bases, we obtain two pencils related by a one-one algebraic correspondence. These pencils are accordingly homographic and projective and the ranges $[P_1]^2$, $[P_2]^2$ are projective.

Again, the pencil of the second order formed by the tangents to a conic is homographic with the range of the second order formed by their points of contact, since, by Chasles' Theorem, the two are equi-anharmonic.

158. Geometrical evidence of homography. It may be asked: when may we assert, from purely geometrical evidence, that the correspondence between two forms is homographic? For if we had to have recourse to analysis every time in order to apply the test whether the connecting relation is of the homographic type, the labour of calculation would in many cases be considerable, and the principle would be of little value in pure geometry.

We shall therefore suppose that our attention is to be confined to what are called algebraic curves or surfaces, that is, curves or surfaces whose equations are rational and integral in the co-ordinates. The conditions (a) that a point shall lie on such a locus, (b) that a straight line or plane shall touch such an envelope, are rational integral algebraic in the co-ordinates of the point, line, or plane. Therefore if a correspondence be established by means of the following processes: (1) taking joins of points or meets of planes, or planes through points and lines or meets of planes and lines; (2) finding intersections of algebraic curves or surfaces with straight lines or with other algebraic curves or surfaces; (3) drawing tangent lines or planes to such algebraic curves or surfaces, or finding points of contact of such tangent lines or planes (note that this includes finding common tangents to two curves or surfaces and also constructing polars): at each step, *provided we nowhere introduce an arbitrary restriction on our choice of alternatives*, an algebraic condition is brought in, which is rational and integral. In the process of elimination no radicals and no transcendental functions can be introduced (for the complete eliminant of two algebraic equations for any variable is known to be a rational integral function of their coefficients). Hence the final relation between the co-ordinates is algebraic and rational.

The above justifies the statements made in Arts. 151, 153 that the co-ordinates of two corresponding points or lines of two projective forms are connected by rational algebraic relations. For clearly the processes of projection fall under the above headings.

We may note in passing that the same type of reasoning will show that any curve obtained from an algebraic curve by processes of this kind is likewise an algebraic curve. Thus the projection of an algebraic curve is an algebraic curve. In particular, the circle being an algebraic curve (its equation referred to rectangular axes through its centre being $x^2+y^2=r^2$), the conic is also an algebraic curve.

Next, as to being certain from geometrical evidence that the correspondence is really one-one. It should be borne in mind that the correspondence must be *intrinsically*, and not *accidentally*, one-one, that is, the fact of its being one-one must depend on the *intrinsic* nature of the curves used, such as their *degree* or *class*, and not on *accidental* characteristics, such as their *position* or *shape*. In this way alone can we be sure that the correspondence is still one-one when imaginary elements are taken into account, and without such assurance we cannot be sure that we are dealing with a homography.

For example the relation

$$x = x'^3$$

is a one-one relation between x, x' so far as real values are concerned, but it is not a homographic relation.

We may describe it geometrically thus:

Take a point P on the axis Ox whose co-ordinate is x. Draw through P a parallel to Oy meeting the straight line $y = x$ at P_1. Through P_1 draw a parallel to Ox meeting the cubic curve $y = x^3$ at P_2. Through P_2 draw a parallel to Oy meeting Ox at P'. P' is the point corresponding to P.

Put in this form the reason why the correspondence is not homographic is geometrically obvious. For although P_1P_2 meets $y = x^3$ in only one *real* point, the curve being of the third degree must be met by any straight line in *three* points. Thus there will be three points P_2 and therefore three points P' corresponding to one point P, but two of these are imaginary.

Again, if a line AB of constant length moves with its extremities on a fixed conic the pencils $O[A]$, $O[B]$, where O is a fixed point on the conic, are not homographic. For the given condition is equivalent geometrically to stating that B is the intersection with the conic of a circle of fixed radius and centre A. This circle has four intersections with the conic, any one of which may be taken for B. Therefore to one ray OA should correspond four rays OB, and it is only by an *arbitrary convention* (to secure continuity of sliding motion) that this number is reduced to unity.

If, however, the conic is a circle, the geometrical conditions that the chord AB is of constant length, and the arc AB is always measured in the same sense, may be expressed in another manner, as follows. Let A_1B_1 be a given position of AB. Given any other position of OA, draw A_1Q parallel to OA to meet the circle again at Q. Then OB is parallel to B_1Q. In this form the correspondence is clearly one-one.

The above will give the reader some notion of the limits within which the application of the principle of one-one correspondence is valid, but rapidity and certainty in recognising these geometrically will be best ensured by the consideration of examples.

159. Every curve of the second degree is a conic. For let O, O' be two points on a curve of the second degree. Draw any ray OP through O: it meets the curve at one other point P, since

O is already on the curve. Join $O'P$. Then if we start from OP, $O'P$ is uniquely determined. Conversely if we start from $O'P$, since O' is already on the curve, $O'P$ meets the curve again at one point only, hence OP is uniquely determined. $O[P]$, $O'[P]$ are therefore homographic pencils. Hence they are projective. Therefore by Art. 41 the locus of P is a conic.

In like manner we can show that every plane curve of the second class is a conic. For let t, t' be two tangents to the curve. On t take any point T. Through T one other tangent p can be drawn to the curve and one only, meeting t' at T'. T, T' are seen to correspond uniquely. Hence $[T]$, $[T']$ are homographic and therefore projective: by Art. 42, TT' envelops a conic.

160. Notation for homography. The symbol $\barwedge$ which was introduced in Art. 24 for "is projective with" will now be extended to homographic forms and be read "is homographic with." This notation will not contradict the previous, since projective forms are homographic.

161. Homographic plane fields. The notion of homography need not, however, be restricted to geometric forms of one dimension. In future, if a correspondence is established between the points of two planes, which correspondence is not limited to particular figures but embraces the aggregate of the points of the planes, we shall speak of the elements of the planes, connected by this relation, as forming corresponding **plane fields,** in the same way that the points of two straight lines may be arranged in corresponding ranges. We have already had examples of such fields; thus we have seen that space or plane perspective, for instance, establish a relation which is not limited to selected points and lines. In fact we have frequently used the term *figure* in the sense here given to *field*, and it will often be convenient still to do so, where no ambiguity is likely to result. A figure, however, is only part of a field, and it is sometimes desirable to have distinct words to denote them.

If two plane fields ϕ, ϕ' correspond point to point in such a manner that the points of a straight line in ϕ correspond to the points of a straight line in ϕ' the fields are said to be collinear, or related by a collineation (*cf.* Art. 3).

If, further, the correspondence between the points of ϕ and ϕ' is one-one and algebraic, the fields are said to be **directly homographic,** or, more briefly, **homographic.** The relation between them is then homography. As a special case, two homographic fields may be in the same plane.

Let x, y be the co-ordinates in the plane of ϕ of a point P of ϕ.

Let x', y' be the co-ordinates in the plane of ϕ' of the corresponding point P' of ϕ'.

Then if the correspondence between x, y and x', y' is to be one-one, x', y', when solved for, must not involve radicals containing x, y, that is, they must be rational functions of x, y. Reducing them to the same denominator we have

$$x' = \frac{P}{R}, \quad y' = \frac{Q}{R} \quad \text{.........................(1)}$$

where P, Q, R are polynomials in x, y.

To the straight line

$$l'x' + m'y' + 1 = 0$$

of the figure ϕ' corresponds the locus

$$l'\frac{P}{R} + m\frac{Q}{R} + 1 = 0 \quad \text{.........................(2)}$$

of the figure ϕ.

This locus (2) is not a straight line unless P, Q, R either reduce to expressions of the first degree in x, y or else have a common factor, such that when it is divided out of P, Q, R, the remaining factor is of the first degree.

In either case equations (1) reduce to the form

$$x' = \frac{A_1x + B_1y + C_1}{A_3x + B_3y + C_3}, \qquad y' = \frac{A_2x + B_2y + C_2}{A_3x + B_3y + C_3} \quad \text{...........(3)}$$

and then the locus (2) becomes the straight line

$$l'(A_1x + B_1y + C_1) + m'(A_2x + B_2y + C_2) + A_3x + B_3y + C_3 = 0,$$

which being reduced to the form

$$lx + my + 1 = 0,$$

gives

$$l = \frac{A_1l' + A_2m' + A_3}{C_1l' + C_2m' + C_3}, \quad m = \frac{B_1l' + B_2m' + B_3}{C_1l' + C_2m' + C_3} \quad \text{........(4)}$$

showing that the line co-ordinates transform according to a similar law.

The equations (3) can be written

$$(A_3x' - A_1)x + (B_3x' - B_1)y + (C_3x' - C_1) = 0,$$

$$(A_3y' - A_2)x + (B_3y' - B_2)y + (C_3y' - C_2) = 0.$$

Solving these for x we find

$$\left.\begin{aligned} x &= \frac{(C_3y'-C_2)(B_3x'-B_1)-(C_3x'-C_1)(B_3y'-B_2)}{(A_3x'-A_1)(B_3y'-B_2)-(A_3y'-A_2)(B_3x'-B_1)} \\ &= \frac{(B_2C_3-B_3C_2)x'+(B_3C_1-B_1C_3)y'+(B_1C_2-B_2C_1)}{(A_2B_3-A_3B_2)x'+(A_3B_1-A_1B_3)y'+(A_1B_2-A_2B_1)} \\ &\text{and similarly} \\ y &= \frac{(C_2A_3-C_3A_2)x'+(C_3A_1-C_1A_3)y'+(C_1A_2-C_2A_1)}{(A_2B_3-A_3B_2)x'+(A_3B_1-A_1B_3)y'+(A_1B_2-A_2B_1)} \end{aligned}\right\} \quad \ldots(5)$$

Equations (5) show that the transformation from x, y to x', y' is of the same type as the transformation from x', y' to x, y. We deduce that to a straight line of ϕ corresponds a straight line of ϕ', and one only, which can be otherwise established by solving back equations (4) for l', m'.

It is clear from the definition that corresponding ranges and corresponding pencils in two homographic fields are themselves homographic.

162. A plane homography is determined by two corresponding tetrads. Let $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ be two *tetrads* or sets of four points in the plane fields ϕ_1, ϕ_2. These tetrads may be arbitrarily given, with the one restriction that no three points in either tetrad are to be collinear. Then a homographic correspondence can be established between ϕ_1 and ϕ_2 as follows.

Let P_1 be any point of ϕ_1. Draw through A_2 a ray A_2P_2 such that

$$A_2\{B_2C_2D_2P_2\} = A_1\{B_1C_1D_1P_1\} \quad \ldots\ldots\ldots\ldots(1)$$

There is only one such ray by Art. 25.

Also draw through B_2 a ray B_2P_2 such that

$$B_2\{A_2C_2D_2P_2\} = B_1\{A_1C_1D_1P_1\} \quad \ldots\ldots\ldots\ldots(2)$$

P_2, being the intersection of A_2P_2, B_2P_2, is determined uniquely when P_1 is given, and conversely. This construction then establishes between ϕ_1 and ϕ_2 a one-one point to point correspondence, which is easily verified to be algebraic.

To prove that it is a homography we have to show that if P_1 describes a straight line, P_2 describes another straight line.

Now from (1) and (2) above

$$A_2[P_2] \barwedge A_1[P_1],$$

A_2B_2 corresponding to A_1B_1, and

$$B_2[P_2] \barwedge B_1[P_1],$$

B_2A_2 corresponding to B_1A_1.

Now if P_1 describes a straight line, $A_1[P_1]$, $B_1[P_1]$ are perspective, A_1B_1 being self-corresponding. Hence

$$A_2[P_2] \barwedge B_2[P_2]$$

and A_2B_2 is self-corresponding: that is, $A_2[P_2]$, $B_2[P_2]$ are perspective and P_2 describes a straight line. The given construction therefore determines a homography.

Also it is the only homography in which A_1, B_1, C_1, D_1 correspond to A_2, B_2, C_2, D_2 respectively. For if P_1, P_2' be corresponding points in any other homography satisfying the given conditions, P_1, P_2' must satisfy the relations

$$A_2\{B_2C_2D_2P_2'\} = A_1\{B_1C_1D_1P_1\} \quad \text{.............(3)}$$
$$B_2\{A_2C_2D_2P_2'\} = B_1\{A_1C_1D_1P_1\} \quad \text{.............(4)}$$

since in a homography corresponding pencils are projective. Comparing (3) and (4) with (1) and (2) we see that $P_2' = P_2$.

The construction cannot fail unless two of the rays of one of the pencils $A_1(B_1C_1D_1)$, $A_2(B_2C_2D_2)$, $B_1(A_1C_1D_1)$, $B_2(A_2C_2D_2)$ coincide, that is, unless three of the points of either tetrad are in one straight line. In this case no homography can exist unless the three corresponding points are also in a straight line. But then the homography is no longer completely determined. For if A_1, B_1, C_1 be points on a straight line p_1 and A_2, B_2, C_2 the corresponding points on a straight line p_2, the triads $A_1B_1C_1$, $A_2B_2C_2$ determine completely the corresponding points of p_1, p_2. If now a fourth point D_1 be given corresponding to a fourth point D_2 and P_1 be any fifth point to which P_2 corresponds, the point in which D_2P_2 meets p_2 corresponds to the point in which D_1P_1 meets p_1 and is known. Therefore D_2P_2 is known, but the position of P_2 on it is indeterminate.

In like manner it may be shown that a homography is determined when four lines of one field, no three of which are concurrent, are made to correspond to four lines of the other field, no three of which are concurrent.

163. Vanishing lines. The equations of the vanishing lines of the homography are easily written down from equations (3) and (5) of Art. 161. For if x', y' are to be infinite we must have

$$A_3x + B_3y + C_3 = 0.$$

This then is the vanishing line of the field ϕ. If x, y are to be infinite, then

$$(A_2B_3 - A_3B_2)x' + (A_3B_1 - A_1B_3)y' + A_1B_2 - A_2B_1 = 0,$$

and this gives the vanishing line of the field ϕ'.

164. Any four coplanar points can be projected into any four coplanar points. Let A_1, B_1, C_1, D_1 be four points, no three of which lie in a straight line, in a plane α_1; and A_2, B_2, C_2, D_2 be four points, no three of which lie in a straight line, in a plane α_2. Through A_1 draw a plane α_3 not coincident with α_1. Project A_2, B_2, C_2, D_2 on to α_3 from a point S on A_1A_2 other than A_2. Let the projected points be A_1, B_3, C_3, D_3.

Let $$(A_1B_1, C_1D_1) = E_1; \quad (A_1B_3, C_3D_3) = E_3.$$

Because the straight lines $A_1B_1E_1$, $A_1B_3E_3$ intersect, B_1, B_3, E_1, E_3 are coplanar. Therefore B_1B_3, E_1E_3 meet at a point U.

Through the line $A_1B_1E_1$ draw a plane α_4 not coincident with α_1. Let the projections of A_1, B_3, C_3, D_3, E_3 from U on to α_4 be A_1, B_1, C_4, D_4, E_1.

The points C_4, D_4, E_1 are collinear, since C_3, D_3, E_3 are collinear. Hence the lines $C_1D_1E_1$, $C_4D_4E_1$ are coplanar. $\therefore$ C_1C_4, D_1D_4 meet at some point V.

Projecting $A_1B_1C_4D_4$ from V on to α_1 we obtain $A_1B_1C_1D_1$. Thus we may pass from A_2, B_2, C_2, D_2 to A_1, B_1, C_1, D_1 by three projections.

It has been assumed that the line A_1A_2 does not lie in α_2; unless the two quadrangles are coplanar, we can always find at least two pairs of which this is true, and one of these may be denoted by A_1, A_2. If however $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are coplanar, let one of them be first projected from any vertex on to another plane, and we have the case already dealt with. Four projections then enable us to pass from A_2, B_2, C_2, D_2 to A_1, B_1, C_1, D_1.

Similarly any four coplanar lines a_1, b_1, c_1, d_1, no three of which pass through a point, can always be projected into any four coplanar lines a_2, b_2, c_2, d_2, no three of which pass through a point. For in this case the four points a_1b_1, b_1c_1, c_1d_1, d_1a_1 are distinct and no three of them are collinear, and the same holds of the four points a_2b_2, b_2c_2, c_2d_2, d_2a_2. These two sets of four points are therefore projective by the first part of the present article, and the lines which join them are likewise projective which proves the proposition.

165. Every plane homography is a projective transformation and conversely. For consider any plane homography. Take two corresponding tetrads such that no three points of each are collinear, and construct a projective transformation in which these are corresponding tetrads. Since both homography and projection preserve cross-ratio constant, the construction given in Art. 162 for finding the point P_2 corresponding to any given point P_1 applies to both the projective and homographic transformations. These two transformations therefore determine the same correspondence between the two fields, or any corresponding figures which are part of them, that is, the given homography is identical with the projective transformation.

Conversely every projective transformation is homographic, for it is a one-one algebraic transformation in which points correspond to points and straight lines to straight lines.

It follows from Art. 162 that two corresponding tetrads of points or lines entirely determine the projective correspondence between two planes.

Two coplanar projective fields with four self-corresponding points, of which no three are collinear, or with four self-corresponding lines, of which no three are concurrent, must therefore be identical.

166. Deductions from the above. If we are given three points A_1, B_1, C_1 on a conic s_1 and three points A_2, B_2, C_2 on a conic s_2 the conic s_1 can always be projected into the conic s_2 and at the same time the three points A_1, B_1, C_1 into the three points A_2, B_2, C_2.

For draw the tangents to s_1 at A_1, B_1 meeting at D_1 and the tangents to s_2 at A_2, B_2 meeting at D_2. Project the four points A_1, B_1, C_1, D_1 into the four points A_2, B_2, C_2, D_2. Then s_1 projects into a conic which touches D_2A_2 at A_2, D_2B_2 at B_2 and passes through C_2. But this conic must be s_2 for two pairs of coincident points and another point determine a conic uniquely.

In like manner if a_1, b_1, c_1 be three tangents to a conic s_1; a_2, b_2, c_2 three tangents to a conic s_2, let d_1 be the chord of contact of a_1b_1, d_2 the chord of contact of a_2b_2. Project $a_1b_1c_1d_1$ into $a_2b_2c_2d_2$. Then s_1 projects into a conic touching a_2 at a_2d_2, b_2 at b_2d_2 and also touching c_2. And this conic can be none other than s_2.

These two results show that two ranges or pencils of the second order can always be actually projected into one another so that any two given triads correspond. Equi-anharmonic ranges of the second order are therefore actually projective, which justifies the name given to them in Chapter VI.

Notice that the projective correspondence between two given ranges of the second order, or two given pencils of the second order, determines entirely the projective relation between the two plane fields to which they belong.

167. Self-corresponding elements of two coplanar projective fields. Consider two coplanar projective fields ϕ, ϕ'. Let O be any point of the plane which is not self-corresponding. To O considered as a point of ϕ' let O_1 correspond in ϕ; to O considered as a point of ϕ let O_2 correspond in ϕ'.

Let p, p' be any pair of corresponding lines of ϕ, ϕ' through O_1, O respectively; then $[p]\barwedge[p']$ and pp' describes a conic u passing through O_1, O. This conic passes through every self-corresponding point P of ϕ, ϕ', since O_1P of ϕ corresponds to OP of ϕ'.

Similarly if q, q' be corresponding lines through O, O_2, qq' describes a conic v passing through O, O_2 and through every self-corresponding point of ϕ, ϕ'.

Hence all self-corresponding points must be intersections of u, v; conversely every intersection of u, v other than O is a self-corresponding point. For, if P be such an intersection, then O_1P, OP of ϕ correspond to OP, O_2P of ϕ' respectively. Their intersections must also correspond, but both are identical with P, which must therefore be self-corresponding.

Since the conics u, v already intersect at O, they have, in general, three other intersections P, Q, R, which are the three self-corresponding points of ϕ, ϕ'.

Similarly, by considering a line x, which is not self-corresponding, and its two correspondents x_1, x_2, we can prove that there are, in general, three self-corresponding lines p, q, r, which are the three common tangents other than x to the conic envelopes s, t of joins of corresponding points on x_1, x and on x, x_2, respectively. These are evidently the lines QR, RP, PQ joining the self-corresponding points P, Q, R.

If the relation between ϕ, ϕ' is real, so that to real points and lines correspond real points and lines respectively, the conics u, v, s, t are all real, and since one intersection O of u and v is real, a second intersection (say P) is always real, the other two, Q and R, being either real or conjugate imaginary. In like manner s, t have always one real common tangent besides x. Thus must be QR, the other two, PQ and PR, being either real or conjugate imaginary.

In general on a self-corresponding line p there are only two self-corresponding points Q, R, the self-corresponding points of the projective ranges formed by corresponding points on p. If however a third self-corresponding point on p exists, then every point of p is self-corresponding and corresponding lines must meet on p. We have then the case of fields in plane perspective, p is the axis of perspective and the self-corresponding point P not on p is the pole of perspective.

In this case OO_1 (or OP) is self-corresponding, so that u becomes the line-pair (OP, p) and v coincides with the same line-pair. The only self-corresponding points on OP, however, are P and the intersection of OP with the axis of perspective p; these may coincide as a special case.

168. Harmonic perspective. If in a plane perspective in which O is the pole and x the axis of perspective, a given pair of corresponding points A, A' are harmonically divided by O and x, the same holds of every other pair of corresponding points.

Denote the field to which A belongs by ϕ and that to which A' belongs by ϕ'. Let P be any other point of the field ϕ. Join AP meeting x at V; then VA' meets OP (*see* Art. 16) at the point P' of ϕ' corresponding to P of ϕ.

If OP, OA meet x at X, B respectively, then by hypothesis $\{OABA'\} = -1$; but $OABA'$ and $OPXP'$ are in perspective from V. Hence $\{OABA'\}=\{OPXP'\}$ so that $\{OPXP'\}= -1$, that is, P, P' are harmonically divided by O and x.

Such a plane perspective is termed **harmonic perspective.**

169. Involutory plane field. It is clear that, if in a harmonic perspective P' is taken at P, then P is at P'. Thus every pair of corresponding points correspond doubly; and it is then obvious that every pair of corresponding lines correspond doubly. The two plane fields are then **involutory.**

Conversely we will now prove that **there can be no involutory plane homography other than harmonic perspective.**

For if, in a homography, a pair of points A, A' correspond doubly, then AA' corresponds to $A'A$ and so is a self-corresponding line x. All pairs of corresponding points on x then correspond doubly and form an involution, of which the double points O, U are self-corresponding points. But this may happen without any pair of points not lying in x corresponding doubly, that is, without the homography being generally involutory. If, however, a second pair of points B, B' lying in another straight line y also correspond doubly, then y is another self-corresponding line, on which pairs of corresponding points form an involution. Further y must meet x at a self-corresponding point. This is either O or U; denote it by O. Let V be the other double point of the involution on y; V is a third self-corresponding point.

Let now p, p' be any pair of corresponding lines meeting x at P, P', y at Q, Q'. Since x is self-corresponding, P and P' are corresponding points, and so are Q, Q'. Hence by the property of the involutions on x and y, $\{OPUP'\} = -1 = \{OQVQ'\}$. Since O is a self-corresponding point of these ranges, PQ, $P'Q'$, UV are concurrent, that is, any two corresponding lines p, p' meet on UV. The homography is therefore a plane perspective with UV as axis. The point O where AA' and BB' meet is then the pole of perspective. And since P, P' are harmonically divided by O and x, the same is true, by Art. 168, of any other pair of corresponding points, and the perspective is a harmonic perspective.

EXAMPLES XA

1. The angles θ, θ' which two lines through a fixed origin make with an initial line are connected by the equation
$$\theta' = \frac{A\theta + B}{C\theta + D}.$$
Explain carefully why the two lines do not describe homographic pencils.

2. If the angles θ, θ' in the last question be connected by the relation
$$\sin\theta' = \frac{A\sin\theta + B}{C\sin\theta + D},$$
show that the lines do not describe homographic pencils.

3. Through the vertex of a flat pencil planes are drawn perpendicular to the rays of the pencil. Show that the axial pencil so formed is homographic with the given flat pencil.

4. A variable circle cuts a fixed circle at a constant angle α and passes through a fixed point O. If the points of intersection of this circle with the fixed circle be P, P', show that the ranges $[P]^2$, $[P']^2$ are homographic, the angle α being measured by the rotation, in a prescribed sense, which brings the tangent to the variable circle upon the tangent to the fixed circle.

5. The co-ordinates of two points on a straight line are connected by the relation
$$\frac{a}{x} + \frac{b}{x'} = \frac{c}{f}.$$
Show that the points describe homographic ranges.

6. A variable conic through four fixed points, two of which lie on a fixed conic s, meets s again at P, P'. O is a fixed point on s. Prove that OP, OP' describe homographic pencils.

7. A variable conic through four points A, B, C, D meets fixed lines through A and B at P and Q. Show that P, Q describe homographic ranges.

8. If in a homographic relation the points (0, 0), (0, 1), (1, 0), (1, 1) in the plane of (x, y) correspond respectively to the points (0, 0), (1, 3), (2, 2), (2, 4) in the plane of (x', y'), show that the point (0, 2) in the plane of (x, y) corresponds to the point $(\frac{4}{3}, 4)$ in the plane of (x', y').

9. In a homography the points of a quadrangle $ABCD$ correspond with themselves in the order $BCDA$. Prove that one of the diagonal points of the quadrangle is a self-corresponding point and the opposite side of the diagonal triangle is a self-corresponding line.

Show further that the other self-corresponding points are always imaginary.

10. Two homographic coplanar figures have the circular lines through O as self-corresponding lines. Show that one of the figures can be brought into plane perspective with the other by a suitable rotation in its own plane about O.

11. Prove that, if the circular points at infinity are self-corresponding points of a homography, corresponding figures in the homographic fields are directly similar, but not similarly situated.

Show further that these are the only homographies in which every circle in the plane transforms into a circle.

12. Given the three self-corresponding points of two projective coplanar figures and a pair of corresponding points, give a construction for the point corresponding to any given point, and also for the vanishing lines.

13. Given three pairs of corresponding points of two homographic plane figures and one of the self-corresponding lines, construct the intersection of the other two self-corresponding lines.

14. Show how to set up a one-one correspondence of a plane into itself such that a circle in the plane is transformed into itself and three assigned points P, Q, R of it into Q, R, P respectively.

Discuss the transformation when Q and R are the circular points at infinity.

15. In two homographic plane figures the lines at infinity correspond : show that the areas of corresponding figures are in a constant ratio.

Deduce that the area of the segment cut off from a parabola by any chord is two-thirds of the area of the triangle formed by the chord and the tangents at its extremities.

16. In two homographic fields (which need not be coplanar), three pairs of corresponding points are given not lying on one pair of corresponding lines, and also one pair of corresponding lines, not incident with any of the given points. Prove that the homography is completely determined, and show how to construct the line of either figure corresponding to a given line of the other.

17. Prove that, in harmonic perspective, any conic with regard to which the pole and axis of perspective are pole and polar transforms into itself.

18. Prove that pairs of corresponding points in a harmonic perspective which are also conjugate for a conic s which is self-corresponding lie on a second conic which has double contact with s.

19. In a harmonic perspective it is given that two conics transform into themselves. Show that there are three possible positions of the pole and axis of perspective and give a method for constructing them.

EXAMPLES X B

1. OUV is the self-corresponding triangle of two coplanar homographic fields ϕ, ϕ'; $UV=5$ inches, $OU=3$ inches, $OV=4$ inches. On two lines OX, OY, making angles of 30° with OU, points A, A', respectively, are taken such that $OA=OA'=2$ inches, A being inside the triangle OUV.

If A, A' are corresponding points of ϕ, ϕ', construct (i) the point of ϕ' corresponding to the middle point of AV in ϕ; (ii) the point of ϕ' corresponding to A' of ϕ; (iii) the vanishing lines of ϕ and ϕ'.

Verify that the last two intersect at the middle point of UV.

2. $ABCD$ is a plane convex quadrilateral: $AB=3$ inches, $AD=DB=4$ inches, $BC=CD=2{\cdot}5$ inches. On AB as side, and on the opposite side to CD, a square $ABC'D'$ is described, the corners A, D' being adjacent.

Obtain the third self-corresponding point of the homography in which A, B, C, D correspond to $ABC'D'$ respectively.

3. Two homographic fields ϕ, ϕ' are referred to rectangular axes Ox, Oy and $O'x'$, $O'y'$ respectively.

The axis of x corresponds to the axis of x' and the points $(1, 1)$ $(2, 3)$, $(4, -1)$ of ϕ correspond to $(0, 2)$ $(4, 1)$ $(3, -2)$ of ϕ'.

Construct (i) the line of ϕ' corresponding to $x+y=0$ of ϕ, (ii) the vanishing line of ϕ, (iii) the points of ϕ, ϕ' corresponding to $(0, 0)$ in the other field.

4. $ABCD$ is a quadrilateral inscribed in a circle of radius 2 inches, the successive arcs AB, BC, CD subtending angles at the centre of 75°, 60°, 135° respectively. $A'B'C'D'$ is a rectangle with $A'B'=1$ inch, $B'C'=2$ inches.

A, B, C, D correspond to A', B', C', D' in two homographic fields ϕ, ϕ'. Construct, in their own figures, (i) the vanishing line of ϕ, (ii) the vanishing line of ϕ', (iii) the point of ϕ' which corresponds to the centre of the circle in ϕ.

5. In a harmonic perspective the pairs of points $(2, 3)$, $(-1, 4)$; $(3, 1)$, $(-2, -2)$ are corresponding pairs. Construct the pole and axis of perspective, the co-ordinates being rectangular.

CHAPTER XI

RECIPROCATION AND INVERSION

170. Reciprocal transformation or correlation. If in the equations (3), (4) and (5) of Art. 161 we interchange l', m' and x', y' we find, writing for shortness

$$\alpha_1 = B_2C_3 - B_3C_2,\ \alpha_2 = B_3C_1 - B_1C_3,\ \alpha_3 = B_1C_2 - B_2C_1,$$

with corresponding meanings for β's and γ's:

$$l' = \frac{A_1x + B_1y + C_1}{A_3x + B_3y + C_3}, \qquad m' = \frac{A_2x + B_2y + C_2}{A_3x + B_3y + C_3},$$

$$l = \frac{A_1x' + A_2y' + A_3}{C_1x' + C_2y' + C_3}, \qquad m = \frac{B_1x' + B_2y' + B_3}{C_1x' + C_2y' + C_3},$$

$$x = \frac{\alpha_1 l' + \alpha_2 m' + \alpha_3}{\gamma_1 l' + \gamma_2 m' + \gamma_3}, \qquad y = \frac{\beta_1 l' + \beta_2 m' + \beta_3}{\gamma_1 l' + \gamma_2 m' + \gamma_3},$$

$$x' = \frac{\alpha_1 l + \beta_1 m + \gamma_1}{\alpha_3 l + \beta_3 m + \gamma_3}, \qquad y' = \frac{\alpha_2 l + \beta_2 m + \gamma_2}{\alpha_3 l + \beta_3 m + \gamma_3}.$$

These equations may be shown as in Art. 161 to be the necessary equations of transformation in any one-one algebraic correspondence of plane fields in which lines correspond to points and points to lines. Clearly any pencil is homographic, and therefore equianharmonic, with the corresponding range. This transformation is therefore of the type discussed in Art. 56.

It is, however, much more general than this transformation; for the transformation by reciprocal polars is limited to fields in the same plane, whereas the present transformation is for any plane fields. Also in the transformation by reciprocal polars the same line p corresponds to the same point P whether P be considered as belonging to one field or to the other. Whereas here, if the fields be taken coplanar and the axes of co-ordinates identical, if we put $x' = x$, $y' = y$ we do not in general obtain $l' = l$ or $m' = m$.

The present transformation is the most general case of a plane reciprocal transformation.

An obvious modification of the reasoning of Art. 162 will show that such a transformation is determined when four points A_1, B_1, C_1, D_1 of one field, no three of which are collinear, are made to correspond to four lines a_2, b_2, c_2, d_2 of the other field, no three of which are concurrent.

For if P_1 correspond to p_2 we have

$$A_1\{B_1C_1D_1P_1\} = a_2\{b_2c_2d_2p_2\},$$
$$B_1\{A_1C_1D_1P_1\} = b_2\{a_2c_2d_2p_2\},$$

which determine a_2p_2 and b_2p_2, and therefore p_2.

And it is easy to show that if P_1 describes a straight line, p_2 passes through a point.

Two such fields will be said to be **reciprocal** or **correlative.** The relation between them may be spoken of as a **reciprocity** or a **correlation.**

171. Two reciprocal transformations are equivalent to a projective transformation. Consider two reciprocal plane fields ϕ_1, ϕ_2, and a third field ϕ_3 reciprocal with ϕ_2. ϕ_1 and ϕ_3 now correspond point by point and line by line and since the correspondence between elements of ϕ_1 and ϕ_2 is one-one and algebraic, and that between elements of ϕ_2 and ϕ_3 is one-one and algebraic, the correspondence between elements of ϕ_1 and ϕ_3 is also one-one and algebraic.

Accordingly the fields ϕ_1 and ϕ_3 are homographic and therefore projective.

172. Any reciprocal transformation is equivalent to a projective transformation and a transformation by reciprocal polars. For let ϕ_1 and ϕ_2 be given reciprocal fields. Let ϕ_3 be the reciprocal polar field of ϕ_2 with regard to any conic in its own plane. Then by the last Article ϕ_1 and ϕ_3 are projective. Thus a projective transformation transforms ϕ_1 to ϕ_3 and the transformation by reciprocal polars transforms ϕ_3 to ϕ_2.

173. Locus of incident points and envelope of incident lines of two coplanar reciprocal fields. If two reciprocal fields ϕ, ϕ' be coplanar, we proceed to find the condition that a point and its corresponding line shall be incident.

If P be a point on its corresponding line p', P being considered as belonging to ϕ, it also lies on its corresponding line when considered as belonging to ϕ'. For let $P = Q'$. Then, since P, *i.e.* Q', lies on p', q passes through P, *i.e.* through Q'. But it should be noted that q is, in general, distinct from p'.

In like manner, if a line $p = q'$ passes through P' it also passes through Q, but P', Q are, in general, distinct.

Such points and lines will be called **incident points and lines** of the correlation.

Consider now an incident point A of ϕ, assuming for the moment that such exist. Let a' be its corresponding line which is then an incident line, and on a' take any point B distinct from A. If B is incident, b' passes through B, and, since A is distinct from B, a' must be distinct from b', otherwise the correspondence would not be one-one.

But AB, which is identical with a', corresponds to $a'b'$ in ϕ', which is identical with B. If then there were two such points B on a', incident and distinct from A, they would correspond to the same line a' in ϕ, which is not possible. Thus there can only be two points on a' which are incident, namely A and one point B, so that it is impossible for an incident line to contain more than two incident points. Thus all the points of the plane cannot be incident points, nor all the lines incident lines.

Let now l be any line whatever, P any point of it, p' the line corresponding in ϕ' to P, P' the meet of l and p'. If to P' in ϕ' corresponds p in ϕ, then since P' lies on p', p passes through P. Each of the points P, P' is thus uniquely determined as soon as the other is known. The ranges $[P]$, $[P']$ are therefore homographic and have in general two self-corresponding points S, T, which may be real coincident or imaginary, and which are such that they lie on their corresponding lines and so are incident points.

The locus k_1 of incident points has thus two intersections with any straight line in the plane. Hence it is a conic by Art. 159.

In like manner through any point M two incident lines can be drawn. The envelope of incident lines is therefore a conic k_2.

The conics k_1, k_2 correspond doubly in the correlation, since a tangent to k_2, being an incident line, corresponds in either field to an incident point, which lies on k_1.

The double character of this correspondence does not, however, extend to the individual elements of the conics.

A point $P = Q'$ (Fig. 54) of k_1 corresponds, in general, to two different incident lines p', q, which are the two tangents from the point to k_2. The tangent $t = u'$ to k_1 at $P = Q'$ corresponds to the point T' of ϕ' where p' touches k_2; and to it corresponds in ϕ the point U where q touches k_2.

In like manner a tangent $p = q'$ to k_2 corresponds, in general, to two different incident points P', Q, which are the intersections of this tangent with k_1. The point of contact $T = U'$ of $p = q'$ with k_2 corresponds to the line t' of ϕ' which touches k_1 at P'; and to it corresponds in ϕ the tangent u to k_1 at Q. It should be noticed that we can here use the same letters without confusion, because p has no necessary relation to p' or T to T', the corresponding elements being p, P'; P, p'; T, t'; t, T'.

If $P = Q'$ comes into coincidence with an intersection X of k_1, k_2, the lines p', q coincide with the tangent x_2 to k_2 at X; thus X corresponds doubly to x_2. But, further, T' and U then both coincide with X, and they correspond to the tangent x_1 to k_1 at

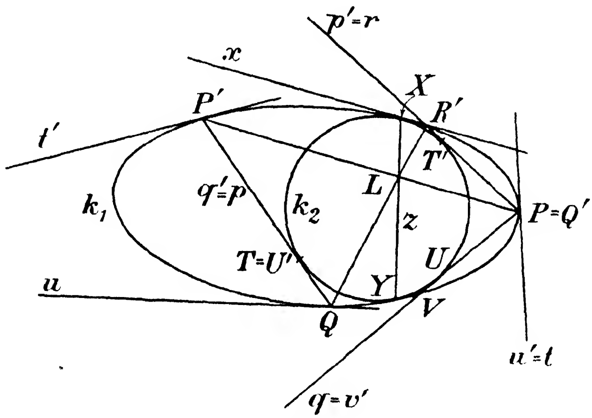


FIG. 54.

$P = Q' = X$; thus X corresponds doubly to x_1. Since the same point cannot correspond to two distinct lines, x_1 must be identical with x_2, and the conics k_1, k_2 touch at any intersection X.

Thus k_1, k_2 either touch at two points X, Y, or else have four-point contact at a point X.

As a special case k_1 may break up into a line-pair, in which case k_2 breaks up into a point-pair. Neither the points of the pair nor the lines of the pair can be incident points or lines, so the point-pair cannot lie upon the line-pair. Each point of k_2 corresponds to the two components of k_1, one in each field.

If the given correlation reduces to a transformation by reciprocal polars, the two conics k_1, k_2 coincide with the base conic, since every point of the base conic lies on its polar and every tangent of this conic passes through its pole. By analogy the conics k_1, k_2 will be termed the **base conics** in the general case.

174. Correspondence of points and tangents on the base conics. In Fig. 54 it is clear that some rule of selection must be given, which enables us to discriminate which end of a chord of k_1 tangent to k_2 is to be treated as P' and which as Q. For the correspondence is not here a double one ; thus in Fig. 54 two such chords are p' and q, their intersection being $P=Q'$. If their other extremities are R', V, it is clear that R' belongs to $p'=r$ and V to $q=v'$. Thus if, in this relation, P corresponds to R', the same point, treated as Q' does not correspond to R' but to a different point V.

Since, in the correlation, homographic ranges and pencils of the second order correspond to homographic pencils and ranges respectively, we have at once

$$[P']^2 \barwedge [p]^2 \equiv [q']^2 \barwedge [Q]^2,$$

so that the ranges $[P']^2$ and $[Q]^2$ are homographic ranges of the second order on the conic k_1. They are not, however, in involution.

But, if P' approaches X, $P'Q$ approaches the tangent at X and Q approaches X. Hence X is a self-corresponding point of the ranges $[P']^2$, $[Q]^2$, and so is Y. Thus the common chord XY of contact of k_1, k_2 is the cross axis of these ranges (Art. 86), and cross-joins such as QR', $P'P$ meet at a point L of XY.

The above will still apply if the base conics have four-point contact at X, since the cross-axis of the ranges is still determinate, being the common tangent at X.

This shows that, once the discrimination has been effected in the case of a single pair of points on k_1, the process of selection is determinate, for, by varying L and keeping P, R' fixed, we can make P', Q represent any required pair of corresponding points and identify the field to which each belongs. Once the points have been identified, the tangents are also identified without ambiguity.

In the first instance, however, we have nothing to guide us in the identification of the original pair of points. But a little consideration will readily show that the choice is really immaterial. For it amounts to interchanging the fields ϕ and ϕ' consistently throughout. This does not affect the base conics since they correspond doubly. The correlation remains the same, the notation only being altered.

175. A correlation is uniquely determined by the base conics. Suppose the base conics k_1, k_2 of a correlation are given, and also the selection rule of Art. 174. This fixes the order of the extremities of two chords such as $P'Q$, $R'S$ (Fig. 55), with the proviso that, if one pair be interchanged, so also must the other pair. Let $A=B'$ be any point of the plane and let the two tangents from this point to k_2 be $p=q'$, $r=s'$, meeting k_1 at P', Q; R', S, respectively.

Now, in ϕ, $A=pr$ and corresponds to $P'R'=a'$ in ϕ'. Similarly $B'=q's'$ in ϕ' and corresponds to $QS=b$ in ϕ. We have thus the

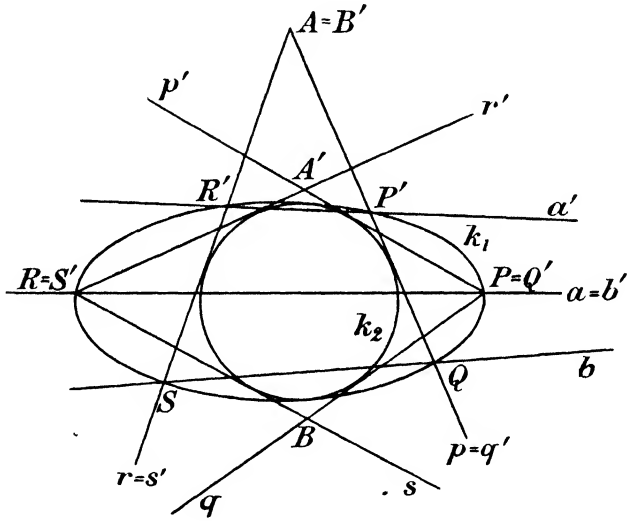


FIG. 55.

two lines which correspond to $A=B'$ when treated as a point of either field.

Again, let $a=b'$ be any given line, meeting s_1 at $P=Q'$ and $R=S'$ (Fig. 55). From these latter points draw tangents p', q and r', s to k_2, the identification following the selection rule. Then $p'r'=A'$, $qs=B$, so that we have constructed the two points corresponding to the line $a=b'$, treated as a line of either field.

Thus the base conics, together with the selection rule, fix the correlation completely and uniquely.

Further, reversal of the selection rule interchanges P' and Q, R' and S, a' and b, A' and B. This simply amounts to interchanging

ϕ and ϕ', as pointed out in the last Article. Thus the correlation is really determined by the base conics alone.

It is to be noted that, although the constructions here given fix the correlation theoretically, as drawing-board methods they are of limited application, since, in general, imaginary elements may enter into them.

The above argument has assumed that the conics k_1 and k_2 were already known to be the base conics of a correlation. If, however, k_1 and k_2 are two conics having either double or four-point contact and arbitrarily given, we may still, provided we definitely allot one of the extremities of a chord of k_1 tangent to k_2 to each the two fields in one particular case and so fix the homographic ranges on k_1, apply the above construction to find the line corresponding to any given point of the plane. In order to show that we then obtain a correlation, it is necessary and sufficient to prove that, if a point A describes a straight line l, its corresponding line a' passes through a fixed point L'.

In such a case the tangents $p=q'$, $r=s'$ (Fig. 55) form an involution of tangents to k_2, of which l is the axis. Since there is a unique correspondence between p and P', r and R', the ranges $[P']^2$, $[R']^2$ are homographic; moreover, if P' is taken at R', p and r are interchanged and R' is at P'. Thus P' and R' correspond doubly and so are mates in an involution on k_1. Hence $P'R'=a'$ passes through a fixed point L', which corresponds in ϕ' to l in ϕ. Similarly $QS=b$ passes through a fixed point which corresponds in ϕ to l in ϕ'. Thus the points of a line correspond to the lines through a point and the relation between the fields is a correlation.

If now the point A lies on a', it must coincide with either P' or R' and so lie on k_1; thus k_1 is the locus of incident points of the correlation. Similarly if a contains A', it must coincide with either p' or r' and so be a tangent to k_2; thus k_2 is the envelope of incident lines of the correlation.

No distinction need be drawn between the cases where the points of contact of k_1, k_2 are separate or coincident. Thus any two conics having either double or four-point contact determine a unique correlation, provided that (1) we assign which is the incident locus k_1 and which the incident envelope k_2; (2) we assign the intersections with k_1 of one tangent to k_2 each to its proper field. By varying the above assignments we obtain four possible correlations from two given conics.

Examples

1. Show that, in the case where k_1, k_2 degenerate into a line-pair and point-pair respectively, the join of the point-pair passes through the meet of the line-pair, and the points of contact X, Y of the general case coincide at this point.

2. Give a construction for the line and point corresponding to a given point and line, when the base conics are a line-pair and point-pair.

176. The self-corresponding triangle. Fig. 55 gives us at once the condition that a point $A = B'$ shall have the same corresponding line in the two fields. For if $a' = b$, then $P' = Q$ and $R' = S$ and the lines $p = q'$, $r = s'$ are tangents to k_1. But they are also tangents to k_2 and so are common tangents to the base conics. There are only two such tangents, namely, those at X, Y (Fig. 54). Moreover, if $P' = Q$ approaches X, $P'Q$ or $p = q'$ approaches the tangent at X. So that the tangent at X is the line corresponding to X in either field. Similarly the tangent at Y is the line corresponding to Y in either field. Calling these tangents x, y, and denoting XY by z, and its pole with regard to either k_1 or k_2 by Z, we have X, Y, Z corresponding to x, y, z in either field. Thus the triangle XYZ corresponds to itself. It should, however, be noted that only one vertex, namely Z, corresponds to the opposite side, the others being incident with their corresponding lines.

177. Reciprocation with respect to a circle. We have already dealt with polar reciprocal figures in Art. 56. There is, however, one special case of polar reciprocation with respect to a conic which is particularly important from the point of view of obtaining what are known as *metrical* properties, that is relations between magnitudes of lengths and angles, especially the latter. This special case arises when the base conic is a circle.

The centre O of the circle with respect to which reciprocal polars are taken is called the **origin of reciprocation** and its radius the **radius of reciprocation.**

Any point P then corresponds to its polar p' with respect to the base circle, and if p' meet OP at N', then $ON'.OP = k^2$, where k is the radius of reciprocation (Art. 54). Similarly a straight line p corresponds to its pole P', so that if N be the foot of the perpendicular from O upon p, P' lies on ON and $ON.OP' = k^2$.

A change in the radius of reciprocation has merely the effect of altering the *scale* of the reciprocal figure. Thus, if a figure ϕ transforms into ϕ' when the radius of reciprocation is k_1 and into ϕ'' when the radius of reciprocation is k_2, the line p transforms in the first case to P' on ON where $OP'.ON=k_1^2$ and in the second case to P'' also on ON, where $OP''.ON=k_2^2$. Thus $OP':OP''=k_1^2:k_2^2$ and the figures ϕ', ϕ'' are similar and similarly situated, O being the centre of similarity, and $k_1^2:k_2^2$ the ratio of similarity. Hence results independent of the scale of the figure are independent of the choice of the radius of reciprocation and the latter may often be chosen so as to have any convenient value, or be left altogether unspecified.

When this last is the case, the reciprocation is specified by the origin O and is often described briefly as reciprocation with respect to O.

Note also that it is not essential that k^2 should be *positive*: if k^2 is negative the construction still leads to a real reciprocal figure, only in this case a line and its reciprocal are on opposite sides of the origin.

The fundamental property of reciprocation with respect to an origin O is the following.

If a, b be any two lines, A', B' their reciprocal points, then OA', OB' are perpendicular to a, b respectively.

Since OA', OB' are drawn either both towards or both away from a and b, $A'OB'$ is *equal* to that angle between a, b which does not include O and supplementary to that which does include O.

Thus: **the angle between any two lines is equal (or supplementary) to the angle subtended by their reciprocal points at the origin of reciprocation.**

Other important properties relate to points at infinity. Thus a point P^∞ reciprocates into the line through O at right angles to OP^∞. The line at infinity reciprocates into O itself.

A circular point at infinity Ω therefore reciprocates into a straight line through O perpendicular to $O\Omega$. But $O\Omega$, being a double ray of a rectangular involution (Art. 141) is at right angles to itself. Thus Ω reciprocates into $O\Omega$ and similarly Ω' reciprocates into $O\Omega'$.

178. Reciprocation of a circle with respect to an excentric origin. Let C (Fig. 56) be the centre of the given circle, O the origin of reciprocation, AB the diameter of the given circle passing through O. The reciprocal curve of the circle will be a conic (Art. 56). Also, by symmetry, one of the axes of this conic will be along AB.

Since Ω, Ω' lie on the circle, $O\Omega$, $O\Omega'$ are tangents to the conic. Hence O is a real focus of the conic, so that AB is the focal axis.

Now C is the pole of the line at infinity with respect to the circle. Hence in the conic the reciprocal c' of C is the polar of O with respect to the conic, that is, the directrix corresponding to O.

Again, the centre C' of the conic is the pole of the line at infinity with respect to the conic, and so is the reciprocal of the polar c of O with respect to the circle. The second focus H is the point such that C' bisects OH.

The tangents from H to the conic are circular lines $H\Omega$, $H\Omega'$

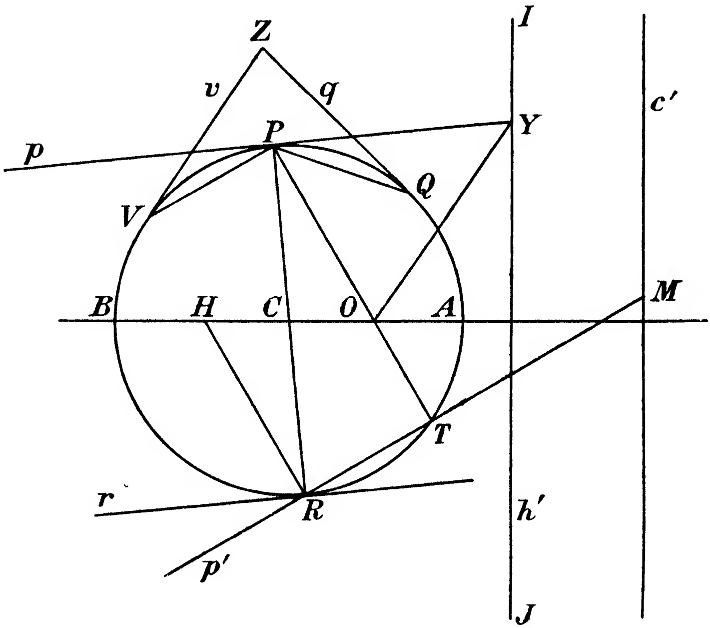


FIG. 56.

which reciprocate into points I, J of the circle, where OI, OJ are perpendicular, and therefore, by the property of circular lines, also parallel, to $H\Omega$, $H\Omega'$ respectively. Accordingly OI, OJ pass through Ω, Ω' respectively, so that IJ and the line at infinity $\Omega\Omega'$ are opposite common chords of the given circle and the line-pair ($O\Omega$, $O\Omega'$), which latter is (Art. 140) identical with the point-circle O. IJ is therefore the radical axis of the given circle and O, and so is a real line h' at right angles to OC. Thus H is the reciprocal of this radical axis, a property which we shall require later.

To find the eccentricity of the conic, reciprocate the tangents a, b at A, B. This gives the vertices A', B' of the focal axis.

Then $OA'=k^2/OA$, $B'O=k^2/BO$, leading to :

$$\text{major axis}=B'O+OA'=k^2\left(\frac{1}{OA}+\frac{1}{BO}\right);$$

$$\text{distance between foci}=OA'+OB'=k^2\left(\frac{1}{OA}-\frac{1}{BO}\right).$$

By division, eccentricity $=\dfrac{BO-OA}{BO+OA}=\dfrac{CO}{CA}$ numerically.

The conic is therefore an ellipse, parabola or hyperbola according as O lies inside, on, or outside the circle.

A very important particular case is that actually shown in Fig. 56. This is when the radius of reciprocation is so chosen that $A'=B$ and $B'=A$. This requires $k^2=OA.OB$, so that for the elliptic case as in the figure, k^2 is negative.

When k^2 is so chosen, the given circle is the circle on the focal axis of the conic as diameter. We see then that a conic and its auxiliary circle are polar reciprocals with respect to an origin at a focus.

Note that in this case C and C' coincide, and also c and c'.

EXAMPLES

1. Given a focus, the corresponding directrix and the eccentricity of a conic show how to construct, without drawing the curve, the two tangents from a given external point O.

Prove that the angle between the tangents from a point O to a parabola is $\cos^{-1}(q/r)$, where r is the distance of O from the focus, and q is the perpendicular distance of O from the directrix.

2. Two circles meet at A and B. Taking A as centre of reciprocation, find the reciprocal theorem of the result that AB is perpendicular to the line joining the centres of the circles.

3. Reciprocate the theorem : circles of constant radius which touch one straight line also touch a parallel line, and their centres lie on the line midway between the two lines.

4. Given a circle centre C and radius r, show how to determine the centre and radius of the circle of reciprocation so that the given circle reciprocates into a rectangular hyperbola of which the real axis is of length $2r$.

5. If the reciprocal of a circle of radius a and centre C is taken with respect to a circle of radius b and centre O, where $OC=c$, find the eccentricity and latus rectum of the conic obtained.

Conics are described with a fixed focus O, their latera recta of the same given length l, and their corresponding directrices tangent to a given conic with focus O and latus rectum l'. Prove that their envelope consists of two conics, each of which has a focus at O, and that the reciprocals of the latera recta of these conics are the sum and difference of the reciprocals of l and l'.

179. Examples of focal properties of conics deduced by reciprocation with respect to a circle. A number of the focal properties discussed in Chapter VIII are very readily deduced from simple properties of the circle by this form of reciprocation. As they give valuable examples of the method, we shall set out the proofs of a few such properties. It will be convenient to assume for k that value which reciprocates the circle into a conic of which it is the auxiliary circle, so that C is the common centre of circle and conic and the foci O, H are symmetrically situated with respect to C.

(*a*) Let v, q (Fig. 56) be any two tangents to the circle, then they are equally inclined to their chord of contact VQ.

Using the fundamental property of reciprocation with regard to a circle (Art. 177), OV', OQ' are equally inclined to $O(v'q')$, where $v'q'$ is the intersection of the tangents at V', Q' to the conic.

Hence the theorem that two tangents to a conic subtend equal or supplementary angles at a focus.

In the figure, which is drawn for the ellipse, the angles ZVQ and ZQV either both include or both exclude O, and we get the case of equal angles.

If O is outside the circle, the points of contact of tangents from O reciprocate into the asymptotes, and divide the arcs of the circle which correspond to the two branches of the curve. The reader will find the discrimination of the two cases for the hyperbola a useful exercise.

(*b*) Let PO (Fig. 56) meet the circle again at T. Then $OP.OT = OA.OB = k^2$. The line through T perpendicular to OT is then a tangent to the conic; it meets the circle again at R, and, since PTR is a right angle, PR is a diameter. Hence by symmetry HR is parallel and equal to PO, and

$$HR.OT = PO.OT = BO.OA = a^2(1-e^2) = b^2,$$

where a is the semi-major and b the semi-minor axis.

We thus obtain the two theorems of Art. 125.

(*c*) Let the tangent p at P meet IJ at Y. By a well-known property of the radical axis of two circles, the lengths of tangents from Y to the given circle and to the point circle O are equal. Thus OPY is an isosceles triangle and PY, OY are equally inclined to OP. Now Y is ph' and reciprocates into $P'H$, so that OY reciprocates into the point at infinity on $P'H$. Similarly OP reciprocates into the point at infinity on p'. Finally PY is p and reciprocates into P'. Hence by the fundamental property OP'

and the parallel through O to $P'H$ are equally inclined to the parallel through O to p', *i.e.* OP', $P'H$ are equally inclined to p' or the tangent and normal bisect the angles between the focal distances.

(*d*) If V, Q are fixed points on the circle, P a variable point, the angle VPQ is constant.

Hence, in the reciprocal figure, the angle between $O(v'p')$ and $O(q'p')$ is constant, *i.e.* the intercept on a variable tangent p' made by two fixed tangents v', q' subtends a constant angle at the focus (Art. 122).

(*e*) The tangent p to the circle at P is perpendicular to the radius vector CP. Therefore OP' is perpendicular to $O(c'p')$, that is, the part of the tangent to a conic intercepted between the point of contact and a directrix subtends a right angle at the corresponding focus (*cf.* Art. 121).

(*f*) ZV, ZQ being two tangents v, q to the circle are equally inclined to CZ. Hence OV', OQ' are equally inclined to $O\{c'(V'Q')\}$, *i.e.* if a chord $V'Q'$ meet the directrix corresponding to focus O at, say X, then OX is a bisector of the angle $V'OQ'$ (*cf.* Art. 121).

(*g*) Two parallel tangents p, r to the circle are a constant distance apart.

But perpendicular distance of O from $p = k^2/OP'$,

and perpendicular distance of O from $r = k^2/R'O$.

Hence $\dfrac{1}{OP'} + \dfrac{1}{R'O} =$ const., where $P'OR'$ is a chord of the conic through O. By taking this chord to be the latus rectum we find this constant $= 2/$(semi-latus rectum), and thus obtain the first theorem of Art. 123.

(*h*) The locus of the intersection Z of tangents v, q to a circle which cut at a given angle α is a concentric circle: this leads at once to the following.

The envelope of a chord $V'Q'$ of a conic which subtends a given angle at the focus is a conic having the same focus and directrix.

Further, in the above VQ touches another fixed circle concentric with the original circle. Hence the locus of the pole $v'q'$ of a chord $V'Q'$ subtending a constant angle at the focus of a conic is a conic having the same focus and directrix.

180. Case where O lies on the circle. When the origin of reciprocation lies on the circle, the line at infinity touches the reciprocal conic, which thus becomes a parabola.

If through O, now on the circle, two perpendicular chords OP, OQ of the circle are drawn, the join PQ passes through the centre C.

We have then : p', q' are perpendicular and $p'q'$ lies on the directrix c', *i.e.* the intersection of two perpendicular tangents to a parabola lies on the directrix (*cf.* Art. 128).

If OA is a diameter of the circle, P any point on the circle, OPA is a right angle. Therefore the reciprocal point $(a'p')$ of AP lies on OP, so that a', p', OP are concurrent. But p' is perpendicular to OP, so that (p', OP) is the foot of the perpendicular from the focus on the tangent p' and this lies on a', which is clearly the tangent at the vertex (*cf.* Art. 127).

There is a well-known theorem on the circle, known as Simson's Theorem, that if L, M, N are the feet of the perpendiculars from a point O of the circle upon the sides f, g, h of a triangle inscribed in the circle, then L, M, N are collinear.

Reciprocating this result with regard to O, we have l' is a line through F' perpendicular to OL, that is, to OF'. Hence if, through the vertices F', G', H' of a triangle circumscribed to a parabola of which O is the focus, we draw lines perpendicular to OF', OG', OH', these are concurrent at some point U. It follows that a circle on UO as diameter passes through F', G', H' or the circumcircle of a triangle formed by three tangents to a parabola passes through the focus (Art. 130).

Examples

1. Show that the vertices of a triangle can be reciprocated into the opposite sides if the origin of reciprocation be taken at the orthocentre.

2. Reciprocate the theorem that the orthocentre of a triangle circumscribing a parabola lies on the directrix, the origin of reciprocation being the orthocentre.

181. Coaxal circles reciprocate into confocal conics. Consider a set of coaxal circles k with real limiting points O, K (Fig. 57). Then O, K are harmonically conjugate with regard to the points P, Q in which any circle of the set meets the line of centres, that is, the polars of O, K with regard to any circle of the set pass through K, O. Also by symmetry these polars are perpendicular to the line of centres.

Therefore O, K have the same polars with regard to all the circles of the coaxal system, and these polars are the perpendiculars to the line of centres through K, O respectively.

Now reciprocate the coaxal circles with regard to any circle centre O. The circles k reciprocate into conics k' having O for a focus.

O reciprocates into the line at infinity, and the line c which is the common polar of O with respect to the coaxal circles reciprocates into a point C' on the line of centres. Thus C' is the common centre of all the conics k'. Since the conics k' have O for a common focus, and C' for a common centre, their second real focus S is also common, so that they are confocal conics. S lies on the line of centres, and $OS=2.OC'$. The reciprocal of S is a line x perpendicular to the line of centres. If it meet this line at X, then

$$OX.OS=OK.OC'.$$

But since $OS=2.OC'$, therefore $OK=2.OX$, or X is the middle point of OK, that is, x is the radical axis.

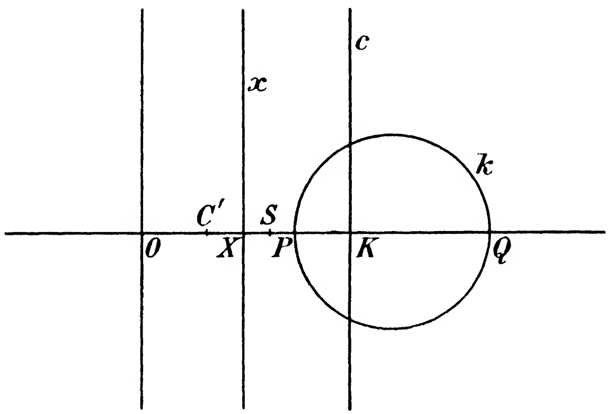


FIG. 57.

Accordingly the two foci are O and the reciprocal of the radical axis.

Conversely, it is easily shown that confocal conics reciprocate with respect to one of the real foci, into a set of coaxal circles with real limiting points.

182. Inversion. Let O be the centre of a circle c, of radius k, P any point in the plane of the circle, P' the point of OP conjugate to P with respect to c, so that

$$OP.OP'=k^2 \quad \text{..........................(1)}$$

The point P' is then said to be the **inverse point** of P with respect to c, and the symmetry of the relation (1) then shows that P is the inverse point of P' with respect to c. The relation between the field formed by the points P and that formed by the points

P' is termed **inversion**, either field being spoken of as the **inverse** of the other with respect to the circle; c, O and k are called the **circle, centre** and **radius** of **inversion** respectively.

Since the polar of P with respect to a circle is perpendicular to OP, it is clear that the inverse point P' is the foot of the perpendicular from O on the polar of P.

Although the correspondence of the points is unique, inversion is not a homography, for a straight line does not correspond to a straight line, but to a circle through the origin.

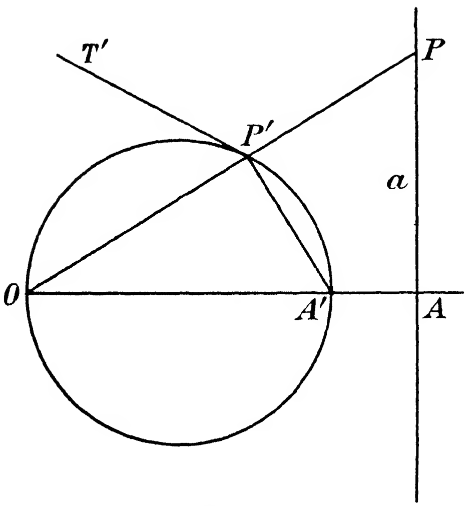


FIG. 58.

To prove this, let a (Fig. 58) be any straight line, A the foot of the perpendicular from O on a, P any other point on a. Let A', P' be the inverse points of A, P. Then $OA.OA' = OP.OP'$, or $OP' : OA' = OA : OP$.

Since the angle at O is common, the triangles AOP, $P'OA'$ are similar, and the angle at P' is equal to the angle at A, and therefore to a right angle. Therefore P' describes a circle on OA' as diameter.

183. The inverse of a circle is, in general, a circle. Let c (Fig. 59) be any circle whose centre is C and radius a. Let P be any point of c, P' its inverse point. Let OP meet c again at Q.

Then: $$OP.OP' = k^2,$$
$$OP.OQ = OC^2 - a^2.$$

Hence $$OP'.OQ = k^2 : (OC^2 - a^2),$$

whence the locus of P' is similar and similarly situated to the locus of Q, O being the centre of similarity, and the ratio of similarity being $k^2/(OC - a^2)$.

Since the locus of Q is the original circle, the locus of P' is also a circle c', whose centre K is in the line OC. If O is outside c, a pair of common tangents to c, c' pass through it.

Thus every circle inverts into another circle, except when the first circle passes through the centre of inversion, when, as in Art. 182, it inverts into a straight line perpendicular to OC.

It should be noted that the centre of the circle c does not invert into the centre of the circle c', so that K is not C'.

It is also important to remember that a *conic* does not, in general, invert into a conic. For, consider the inverse s' of a conic s, and the intersections of s' with a straight line t'. This line t' inverts into a circle t, which has four real or imaginary intersections with the conic s. The inverse points of these four intersections are the

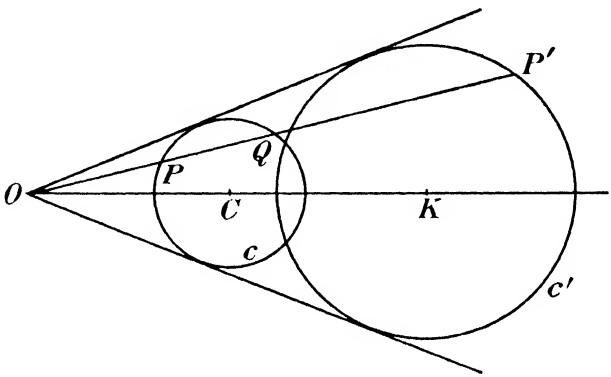


FIG. 59.

intersections of t' and s'. These latter are thus four in number, so that s' is, in general, a quartic curve.

EXAMPLE

Given a point O and a circle centre C and radius r, show how to determine the radius of a circle of inversion, centre O, so that the circle, centre C, inverts into another of radius $2r$, and find the position of the centre of this inverse circle.

184. Circles through two inverse points. Let P, P' be any two inverse points with respect to a circle v of centre O and radius k, and let c be any other circle passing through P and P'.

Let Q be an intersection of v and c. Then $OQ = k$, the radius of inversion, and $OP.OP' = OQ^2$.

Hence, by a well-known property of the circle, OQ is a tangent from O to the circle c, and the circles c, v cut at right angles.

Thus **any circle through two inverse points is orthogonal to the circle of inversion.**

Conversely, if two points P, P' are such that two circles c_1, c_2

through P, P' are orthogonal to the circle of inversion v, then P, P' must be inverse points.

Let Q_1 be an intersection of v and c_1. Since these cut at right angles, OQ_1 is tangent to c_1 at Q_1.

Join OP and produce it to meet c_1 again at R_1. Then by a well-known property of the circle, $OP.OR_1 = OQ^2 = k^2$. R_1 is therefore the inverse point of P with respect to v.

Similarly, if OP meet c_2 again at R_2, R_2 is the inverse point of P with respect to v, and so is identical with R_1, which is thus an intersection, other than P, of the circles c_1, c_2. Since, by hypothesis, this second intersection is P', P' is identical with R_1 and so must be the inverse point of P.

It follows that, if two circles through P, P' are orthogonal to the circle of inversion, every circle through P, P' will be orthogonal to this circle.

EXAMPLES

1. The limiting points of a set of coaxal circles are inverse points with respect to every circle of the set.

2. If the radius of the circle of inversion is made infinite, so that the accessible part of the circle is a straight line, show that any two inverse points are images of one another in the straight line.

3. Prove that a circle passing through two inverse points inverts into itself.

185. Homography on circular bases is preserved by inversion. Although inversion, considered as a relation between plane fields, is not homògraphic, yet it preserves homographic properties as regards ranges on corresponding circles (including straight lines as a particular case).

Referring to Fig. 59, it is clear that, since P', Q are corresponding points in a similarity, and a similarity is a particular case of a plane perspective and therefore of a homography, we must have

$$[Q]^2 \barwedge [P']^2.$$

But, since OPQ is a chord of the circle c through a fixed point, P, Q are mates in an involution on the circle.

Hence $[P]^2 \barwedge [Q]^2$ and $[P]^2 \barwedge [P']^2$.

Thus **a range of the second order on a circle inverts into a homographic range of the second order on the inverse circle.**

The case of a range on a straight line follows immediately from Fig. 58; here $[P] \barwedge O[P] \barwedge [P']^2$.

Since straight lines do not invert into straight lines, this property is not applicable to pencils.

If now we have, in any figure, two homographic ranges $[P_1]^2$, $[P_2]^2$ on circular (or rectilineal) bases, then the inverse ranges $[P_1']^2$, $[P_2']^2$ are such that

$$[P_1']^2 \barwedge [P_1]^2, \ [P_2']^2 \barwedge [P_2]^2$$

and since $[P_1]^2 \barwedge [P_2]^2$, therefore $[P_1']^2 \barwedge [P_2']^2$, or: **homographic ranges on circular bases invert into homographic ranges on the inverse bases.**

186. Angles invert into equal angles. In Fig. 58, let the tangent at P' to the locus of P' be drawn, as $P'T'$.

Then the angle $T'P'O$ is equal to $OA'P'$ in the alternate segment, and this again, by the similarity of the triangles AOP, $P'OA'$, is equal to OPA.

The line a and its inverse circle a' therefore make equal angles with OPP', but on opposite sides of it. By considering small elements it is easily seen that the same holds good of any two corresponding curves.

If now we have two curves s, t intersecting at any angle at P, and the two corresponding curves s', t' intersecting at the corresponding point P', then s, t make with OP, on one side of $OP'P$, angles equal to those which s', t' make with OP' on the other side of $OP'P$. By subtraction, the angle between s, t at P is equal to the angle between s', t' at P'.

Inversion is therefore what is usually termed a **conformal transformation,** that is, any elementary *small* figure inverts into a similar figure. It will be found, however, that they are oppositely, and not directly, similar, that is, they cannot be brought into similar situation by a relative displacement in their plane, but one of them must be turned over as a preliminary.

Note that, as a particular case of the theorem of the present article, an orthogonal intersection of two curves inverts into an orthogonal intersection of the two inverse curves.

Examples

1. Describe, with diagrams, the possible geometrical configurations which may be obtained by inverting the plane figure formed by two parallel straight lines and a line meeting them at right angles, with respect to a circle in the plane.

Three circles touch one another in pairs. Prove that the circle through the three points of contact cuts the three circles orthogonally.

2. Show that if σ, σ' are two very small corresponding areas at corresponding points P, P' in inverse fields, then in the limit $\sigma : \sigma' = OP^4 : k^4$.

187. Inversion of coaxal circles. Let c_1, c_2, c_3, etc., be a set of concentric circles, whose common centre is C. All straight lines l through C are then orthogonal to the circles c.

Now invert the field with regard to any given point O. The straight lines l invert into circles l' passing through O and through the inverse point C' of C, that is, into coaxal circles with real intersections O, C'.

The circles c then invert into circles c' which are orthogonal to the circles l' and therefore, by Art. 113, form a set of coaxal circles with imaginary intersections but real limiting points.

One of these limiting points is clearly C', since C is the point-circle of the set c, and a point-circle must invert into a point-circle.

To find the second limiting point, we note that, if c^{∞} is the circle centre C, of infinite radius, all the points of c^{∞} invert into O, which is therefore a point circle inverse to c^{∞}.

Thus **a set of concentric circles invert into a set of coaxal circles having for real limiting points the centre of inversion and the inverse point of the common centre.**

Conversely, if we invert such a set of coaxal circles with respect to one of their limiting points, we obtain a set of concentric circles. For the coaxal circles through the limiting points are orthogonal to the given coaxal circles, and invert into straight lines through the inverse of the second limiting point, which straight lines are orthogonal to every one of the new set of circles. But a circle which is orthogonal to a pencil of rays through a point has that point for centre; the required result follows.

Note also that, since coaxal circles are circles through two fixed points (real or imaginary), coaxal circles necessarily invert into coaxal circles, whatever the centre of inversion.

We may therefore regard concentric circles as a particular case of coaxal circles. In this case the (ordinary) common points A, B of the circles coincide with Ω, Ω', so that the circles touch at Ω, Ω', as we should expect, and the common radical axis is the line at infinity.

Examples

1. Prove that two circles of a given coaxal system touch an arbitrary fixed straight line, and also that they are inverse with respect to that circle of the system which has its centre on the given straight line.

2. If a circle touches two given circles c_1, c_2, the points of contact are concyclic with the limiting points of the coaxal system determined by c_1, c_2.

3. If two variable circles touch one another, and also each of two given

circles c_1, c_2, show that the locus of their mutual point of contact is a circle coaxal with c_1, c_2.

4. Two given coplanar circles a, b have centres A, B; the inverse of a point P_1 with respect to a is P and the inverse of P with respect to b is P_2. Prove that, when P_1 describes a circle or a line, the corresponding point P_2 describes either a circle or a line; and show that, in the correspondence between the fields $[P_1]$, $[P_2]$, the system of coaxal circles with limiting points A, P_1 corresponds to the system of coaxal circles with limiting points B, P_2.

5. Invert the theorem: the locus of centres of circles which touch two concentric circles is two concentric circles.

188. Inverse figures invert into inverse figures. Let ϕ, ϕ' be two figures which are inverse with regard to a circle k.

Invert both ϕ and ϕ' with respect to another circle s. We obtain two new figures ϕ_1 and ϕ_1'; the circle k inverts, in general, into a circle k_1.

Let P, P' be any two corresponding points of ϕ, ϕ' and let their inverses with regard to s be P_1, P_1'.

Let c_1 be any circle through P_1, P_1'. This corresponds, in the inversion with regard to s, to a circle c passing through P, P'.

By Art. 184, c and k cut orthogonally. Hence, by Art. 186, c_1 and k_1 cut orthogonally. Thus any circle through P_1, P_1' cuts k_1 orthogonally, and, by Art. 184, P_1, P_1' are inverse points with regard to k_1. Since P_1, P_1' are an arbitrary pair of corresponding points, the figures ϕ_1, ϕ_1' are inverse with regard to k_1.

Thus two inverse figures invert into two inverse figures, the circles of inversion being inverted into one another.

EXAMPLE

Prove that two fields inverse with respect to a circle can be inverted into two fields which are the reflection of one another in a straight line.

EXAMPLES XIA

1. Show that a correlation projects into a correlation, and the base conics into base conics.

2. In two coplanar reciprocal fields ϕ_1, ϕ_2 the lines a_2, b_2, c_2 in ϕ_2 which correspond to the vertices A_1, B_1, C_1 of a triangle in ϕ_1 coincide respectively with the opposite sides B_1C_1, C_1A_1, A_1B_1. Show that every point of the plane corresponds to the same line in either field.

3. Show that the only correlation in which every point corresponds to the same line in either field is the transformation by reciprocal polars.

4. Discuss the characteristics of the correlation in which the base conics are concentric circles centre O. Show that the two lines which correspond to any point P are equally inclined to OP and that they intersect on OP

at a point P' which lies on the polar of P with respect to the circle k_1 which is the locus of incident points.

Prove also that the angle between the two lines corresponding to P is constant and equal to the angle between the tangents to k_2 from any point of k_1. Hence show that, if the correlation is real, k_2 must be interior to k_1.

5. By projecting the circular points into any given points, deduce the results corresponding to those of Ex. 4 in the case of a general correlation.

6. In a correlation three lines a, b, c through a vertex U correspond to three points A', B', C' on a line u' through U. Obtain a construction, using the ruler only, for the point V' corresponding to the line $v=u'$, and also for the line t corresponding to the point $T'=U$.

7. Reciprocate the theorem, that the angle in a segment of a circle is constant, with respect to the point at which one of the arms of the angle meets the circle, and hence show that if through the focus O of a parabola a line OP is drawn to meet a variable tangent to the parabola at P and making a fixed angle with this tangent, the locus of P is a fixed tangent to the parabola.

8. From the result that a chord of a circle which touches a concentric circle subtends a constant angle at the common centre deduce the following theorem.

If there be two conics having the same focus and corresponding directrix and if two tangents to one conic meet on the other conic and intersect the directrix at XY, then XY subtends a fixed angle at the focus.

9. If from a fixed point tangents are drawn to a series of concentric circles, find the locus of their points of contact. Hence obtain the following theorem by reciprocation. A fixed line meets a number of conics which have the same focus and corresponding directrix; then the envelope of the tangents at the points of intersection is a conic touching the fixed line and also the common directrix.

Hence find the envelope of the asymptotes of conics having a common focus and directrix.

10. If the two common tangents to two conics having a common focus O intersect at A and through A a line APQ be drawn meeting one conic at P and the other at Q, then, if the tangents at P and Q meet at R, OR is a bisector of the angle POQ.

[Reciprocate the theorem that tangents to two circles from a point on their radical axis are equally inclined to the line joining their points of contact.]

11. A variable chord of a rectangular hyperbola subtends a right angle at a fixed point O. By reciprocation with respect to a circle having O as centre show that the locus of the foot of the perpendicular from O on the chord is a straight line.

12. Prove that the polar reciprocal of a circle, taken with regard to a rectangular hyperbola, is a conic of which the centre of the rectangular hyperbola is a focus.

13. A conic is inscribed in a triangle ABC and has one focus at the circumcentre. Prove that the other real focus is the orthocentre.

14. Through a point O in the plane of a circle perpendicular rays OP, OQ are drawn to the circle. Show that PQ touches a fixed conic with a focus at O.

[Reciprocate the property of the orthoptic circle.]

15. Prove that if through a fixed point O inside a circle a straight line be drawn, meeting the circle at P and Q, the sum of the reciprocals of the perpendiculars from O on the tangents at P, Q is constant.

Obtain from this the theorem that the sum of the focal distances in an ellipse is constant.

16. Show that if ABC is a triangle and O any point in its plane, and if through O perpendiculars OP, OQ, OR are drawn to OA, OB, OC to meet BC, CA, AB respectively at P, Q, R, then PQR are collinear.

17. s' is the reciprocal of a curve s with respect to a point O, and P' is the point of s' corresponding to the tangent to s at P. If ρ, ρ' are the radii of curvature of s, s' at P, P' respectively, prove that $\rho\rho' = k^2 \operatorname{cosec}^3 \phi$, where k is the radius of reciprocation, and ϕ is the angle between OP and the tangent to s at P.

Illustrate this result by using it to find the radius of curvature at a point of a conic.

18. A circle c touches an ellipse k at P, and has its centre O on the orthoptic circle of k; the other common tangents q, r of c and k touch the circle c at Q, R respectively. By reciprocation with regard to c, show that there is one rectangular hyperbola which touches k at P and passes through Q and R; and that its asymptotes are parallel to the tangents from O to k.

Prove also that chords of this hyperbola which subtend right angles at O touch a certain parabola whose focus is O.

19. Prove by reciprocation the following.

A variable tangent p to a conic having S as focus meets two fixed tangents a and b at P and Q respectively; a line q through Q meets a at a point L such that the angle PSL is a constant angle. Show that the envelope of q is a conic having S for focus and touching a and b.

20. Prove that the reciprocal of a parabola k with respect to a circle c, whose centre O lies on k, is another parabola k', which touches at O the perpendicular from O to the axis of k, and has its axis parallel to the normal at O to k.

Show also that the directrix of k' is the reciprocal, with respect to c, of the Frégier point of O with respect to k.

If c has its centre O at an extremity of the latus rectum of k, prove that the focus of k' lies on the tangent at O to k.

21. Reciprocate, with respect to a circle with centre at a focus, the property that confocal conics intersect at right angles.

22. If one conic s is its own polar reciprocal for another conic t then the conic t is its own polar reciprocal for the conic s.

[Show that the conics have double contact at A, B and that if C be the common pole of AB then if a ray through C meet t at U, T the tangent at U is the polar of T with regard to s.]

23. Show that given two points A, B of a conic s, a conic t can be found touching s at A, B and such that s is its own polar reciprocal with regard to t.

24. If a figure be inverted with regard to any origin, show that an involution on a circle inverts into an involution on the corresponding circle.

25. Invert the theorem: tangents to a fixed circle cut a fixed concentric circle at a constant angle, the centre of inversion being a point in the circumference of the second circle.

26. Prove that the circles which pass through a given point L and cut a

fixed circle c at a given angle α all touch a second fixed circle k; and that L is a limiting point of the coaxal system determined by c and k.

Show also how to construct the circle k.

27. PQ is a common tangent to two circles and L, M are the limiting points of the set of coaxal circles determined by the two circles. Prove that PLQ, PMQ are right angles.

Transform this result by inversion, (i) from L, (ii) from M.

If PR, QS are chords of the circles passing through L, prove that RS is also a common tangent.

28. P_1 and P_2 are the inverses of a point P with respect to two different circles k_1 and k_2 respectively. If the inverse of P_1 with respect to k_2 coincides with the inverse of P_2 with respect to k_1, show that k_1 and k_2 are orthogonal.

EXAMPLES XIb

[The axes of co-ordinates are to be taken rectangular.]

1. The points $(3, 1)$, $(-3, 0)$, $(1, 2)$, $(1, -2)$ correspond to the lines $x=3$, $x=-3$, $y=2$, $y=-2$ in a correlation.

Find (i) the other incident points on the four given lines; (ii) the other incident lines through the four given points. Hence construct the base-conics.

2. With the data of Ex. 1, construct (i) the lines corresponding to the origin; (ii) the points corresponding to the line at infinity.

3. The base-conics of a correlation are the hyperbola

$$x^2-3y^2=1$$

and its auxiliary circle. Construct the conic corresponding to the conjugate hyperbola.

4. The circles $(x-2)^2+y^2=9$, $(x+4)^2+y^2=1$ are reciprocated with respect to the circle $x^2+(y-1)^2=1$.

Without drawing the reciprocal curves of the first two circles, find their four intersections.

5. Two circles centres A and B have radii 1 inch and 2 inches respectively, and $AB=4$ inches. Find (i) a circle with respect to which they reciprocate into confocal conics, (ii) the four intersections of these conics, (iii) the tangents at these four intersections.

6. Two circles centres A and B have radii 3 and 4 inches respectively, and $AB=2$ inches. Find a centre of inversion which inverts the above circles simultaneously into straight lines. If P, Q are two points such that $AP=7$ inches, $AQ=1{\cdot}5$ inches, and the angles BAP, BAQ are 30°, 135° respectively, construct the circles orthogonal to the given circles and passing through P, Q respectively. Construct also the circles inverse to the last found circles.

7. A circle passes through the point P $(3, 4)$, and touches the axis of x at the origin O of co-ordinates. If its inverse circle passes through the point P' $(2, -1)$ inverse to P, and the chord of this inverse circle corresponding to OP is of length 4 units, construct the circle of inversion.

CHAPTER XII

HOMOGRAPHIC PLANE FORMS OF THE SECOND ORDER

189. Construction of coplanar homographic forms of the second order. In what follows we shall extend the definition of incident forms, already given for projective ranges and flat pencils in Art. 24, to cover all homographic unlike forms such that each element of one form is incident with the corresponding element of the other form. Thus the pencil of the second order formed by the tangents to a conic is incident with the range of the second order formed by their points of contact, and a range of the second order

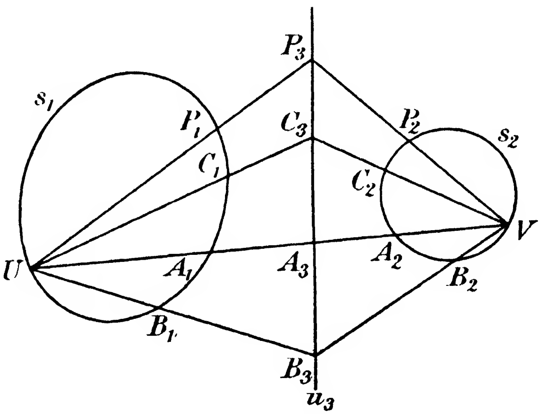


FIG. 60.

is incident with the flat pencil determined by it at any point of its conic base.

Let two projective ranges of the second order on two conics s_1 and s_2 be given by two corresponding triads $A_1B_1C_1$, $A_2B_2C_2$ (Fig. 60). Corresponding points of the two ranges may be constructed as follows.

Join A_1A_2 meeting s_1 again at U and s_2 again at V. Let UB_1, VB_2 meet at B_3 and UC_1, VC_2 meet at C_3. Join $B_3C_3 = u_3$ meeting UV at A_3. Then if P_1 be any point on s_1 and UP_1 meets u_3 at

P_3 and VP_3 meets s_2 again at P_2, the ranges of the second order $[P_1]^2$, $[P_2]^2$ are homographic and they have $A_1B_1C_1$, $A_2B_2C_2$ for corresponding triads. They are therefore the ranges required.

If one of the ranges, say $A_2B_2C_2$, is of the first order, a similar construction holds, but this time V may be taken any point on A_1A_2.

Similarly if two pencils of second order about conics s_1, s_2 be given by the corresponding triads $a_1b_1c_1$, $a_2b_2c_2$ (Fig. 61), then from $A(=a_1a_2)$ draw the other two tangents u, v to s_1, s_2. Let

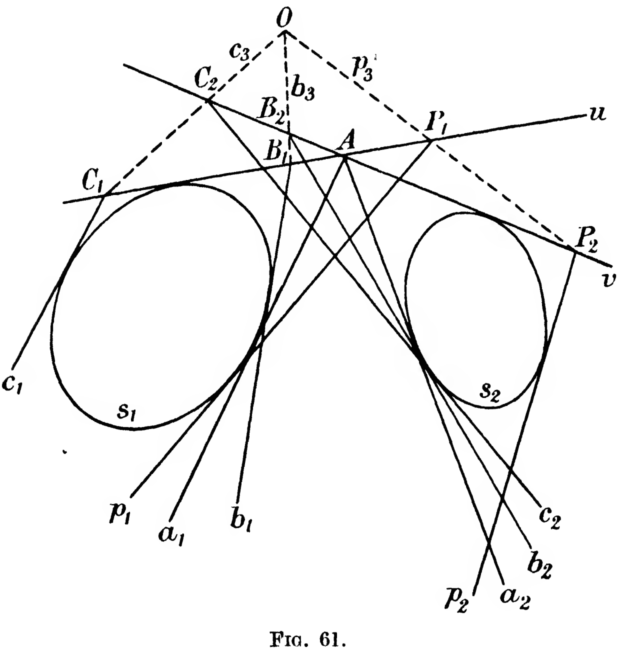


FIG. 61.

$ub_1=B_1$, $uc_1=C_1$, $vb_2=B_2$, $vc_2=C_2$. Let $B_1B_2=b_3$, $C_1C_2=c_3$ and let O be their intersection; let $OA=a_3$. Then if p_1 be any tangent to s_1 meeting u at P_1, and if OP_1 be joined to meet v at P_2 and p_2 be the other tangent from P_2 to s_2, the pencils of the second order $[p_1]^2$, $[p_2]^2$ are homographic and have $a_1b_1c_1$, $a_2b_2c_2$ for corresponding triads.

A similar construction holds if the pencil $[p_2]$ is of the first order, only now v may be taken any line through A and p_2 is joined to the vertex of the pencil $[p_2]$ instead of being drawn tangent to a conic.

If the given forms are unlike, say a range and a pencil of second order, we can correlate as above the given range of the second order with the range formed by the points of contact of the given pencil of second order. In this way the two original forms are geometrically connected.

190. Number of self-corresponding elements of homographic forms of first and second order, not on the same base. Clearly a range of the first order and a range of the second order homographic with the first range cannot have more than two self-corresponding elements since a straight line meets a conic in two points only. They *may* have two self-corresponding elements, for if we take a flat pencil whose vertex is on a conic, the homographic ranges determined by this pencil on the conic and on any straight line have the intersections of the straight line and conic for self-corresponding points.

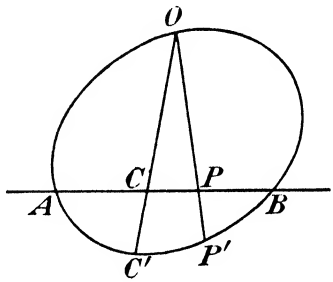


FIG. 62.

Conversely if such ranges have two self-corresponding points, say A, B, the lines joining their corresponding points pass through a vertex O lying on the base of the range of second order. For let C, C' be corresponding points on the straight line and conic respectively (Fig. 62). Join CC' meeting the conic again at O. Then if P, P' are on a line through O, the ranges $[P]$, $[P']^2$ are projective. But they are determined by the same triads ABC, ABC' as the original ranges. They are therefore identical with these ranges.

In like manner if a pencil of the first order be homographic with a pencil of the second order, they may have two self-corresponding elements, namely the tangents from the vertex of the first pencil to the conic which is the base of the second pencil, but they cannot have more. Also, when they do have two self-corresponding lines, we can show, by reasoning similar to that used for the ranges, that corresponding lines intersect on a tangent to the conic which is the base of the pencil of second order.

191. Two homographic forms of the second order cannot have more than three self-corresponding elements. Two homographic ranges of the second order have their bases s_1 and s_2

intersecting in four points, but they cannot have more than three self-corresponding points.

In the first place we will show that they may have three self-corresponding points. For let O, A, B, C (Fig. 63 (a)) be the four intersections of the conics s_1, s_2; through O draw any ray to meet s_1 again at P and s_2 again at P', then the ranges $[P]^2$, $[P']^2$ are homographic and they have the points A, B, C self-corresponding.

In the second place two such ranges cannot have more than three self-corresponding points. For let A, B, C be self-corresponding points. Then the self-corresponding triad ABC determines the correspondence between the two ranges uniquely. Now the ranges

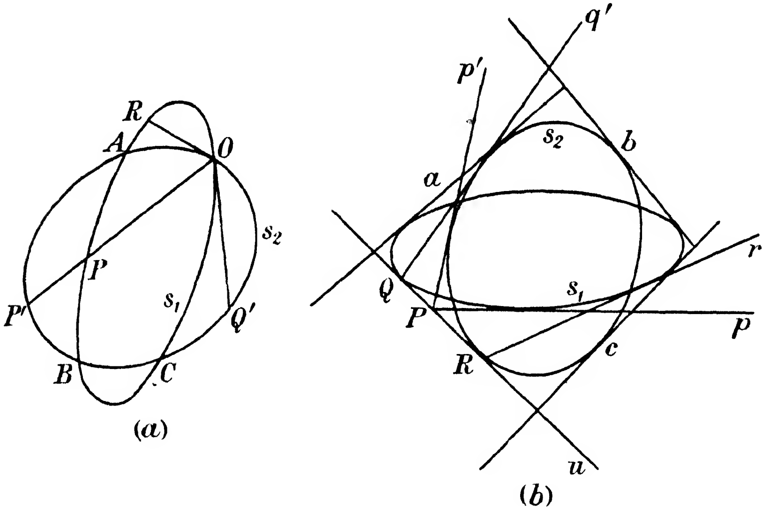


FIG. 63.

described on the conics by a ray OPP' through O satisfy the given conditions, since they have A, B, C for self-corresponding points. The given ranges $[P]^2$, $[P']^2$ are therefore such that the join PP' of corresponding points passes through O. If these ranges have a fourth self-corresponding point this can only be O. But the conics do not touch at O, since they already have three other intersections A, B, C. Thus if P be at O, P' is at Q' where the tangent at O to s_1 meets s_2, and if P' be at O, P is at R where the tangent at O to s_2 meets s_1. Therefore O is not in general self-corresponding.

If the conics touch at O, O is indeed self-corresponding, but the conics, having already two coincident intersections at O, can have only two other distinct intersections; so that in any case there are not more than three self-corresponding points.

In like manner two homographic pencils of tangents to two conics s_1, s_2 (Fig. 63 (*b*)) can have at most three self-corresponding elements, namely three of the common tangents to s_1, s_2. For, if a, b, c be these common tangents, they determine, as in the case of ranges, the relation between corresponding tangents of the pencils, namely, that two such corresponding tangents p, p' meet at P on the fourth common tangent u to s_1, s_2. But u is not self-corresponding; for if p is taken coincident with u, P is at the point of contact Q of u with s_1 and the second tangent q' from Q to s_2 is not coincident with u, unless s_1 and s_2 touch u at the same point. But in this case two of the common tangents coincide with u and there are only two others remaining.

In the above, no distinction has been made between real and imaginary intersections or common tangents. We shall follow this practice in future, except where the contrary is explicitly stated.

192. If a form of the first order has more than three elements incident with their corresponding elements of a homographic form of the second order, the two forms are altogether incident. In the case of two forms of the first order, if more than *two* elements are incident, the forms are incident. Thus, if three rays of a flat pencil, a_1, b_1, c_1, pass through the corresponding points A_2, B_2, C_2 of a homographic range on a straight line u, the range determined by the pencil upon u has three self-corresponding points with the original range and so coincides with it.

In the case of forms of the second order, however, this no longer applies, because the points in which a pencil of the second order meets any straight line do not form a range homographic with the pencil, unless the straight line happen to be a tangent to the conic which is the base of the pencil. The student can easily convince himself of this by reversing the process, when he will find that, although to each tangent to the conic corresponds only *one* point of the straight line, to each point of the straight line correspond *two* tangents to the conic. The correspondence is therefore not one-one.

Consider a range of the second order $[P]^2$ on a conic s homographic with a pencil $[p']$ of the first order whose vertex is U. Take a vertex O on s and join $OP = p$. The pencil $[p]$ is of the first order and $[p] \barwedge [P]^2 \barwedge [p']$. The locus of $Q = pp'$ is a conic t passing through O and the vertex of $[p']$. The conics s, t have therefore three other intersections besides O, of which at least one is real if s, t and O are all real, since by Art. 138 imaginary intersections occur in pairs and O is already one real intersection. But at an intersection of s, t the ray p' passes through its corresponding point P, and, conversely, if p' pass through P, P is an intersection of s, t. Hence there are three pairs of corresponding elements incident, of which one pair is always real.

This holds even if O be on its corresponding ray, for then, if P be at O, OP is tangent to the conic t (since it corresponds to the join of the vertices). But OP is also tangent to the conic s. Thus s, t touch at O and have only two other intersections.

If then there were a fourth pair of corresponding elements incident, the conics s, t would have five points common and would coincide; the vertex of $[p']$, which lies on t would then also lie on s, and every pair of corresponding elements would be incident.

Reciprocating this theorem we obtain the corresponding theorem for a range of the first order and a pencil of the second. The proof is precisely similar to the one above if we interchange terms according to the rules given in Art. 57. The result runs:

If a pencil of the second order and a range of the first order be homographic, then, in general, three pairs of corresponding elements are incident, of which at least one pair are real. If more than three pairs are incident, the two forms are altogether incident and the base of the range touches the base of the pencil.

193. The product of two homographic pencils of the first and of the second order respectively is a cubic. If we form the product of two coplanar homographic pencils $[p]$, $[p']^2$, we obtain a curve in the plane. Draw any straight line u in the plane. This meets $[p]$ in a homographic range of the first order $[P]$. Then, if $Q = pp'$, a point P of u is on the locus of Q if it lies on its corresponding line p'. By the last Article there are three such points P on every line u, of which one at least is real. The locus is therefore one of the third degree, or, as it is called shortly, a **cubic.**

194. The vertex of the pencil of the first order is a double point on the cubic. Let O be the vertex of the pencil of the first order, s the conic which is the base of the pencil of the second order. Let u', v' be the two tangents from O to the conic, u, v their corresponding rays through O.

Then O appears *twice* on the locus, once as uu' and once as vv'. Also corresponding to these two interpretations of O there is a different tangent to the curve. For if p' approaches u', p approaches u and the point $Q = pp'$ approaches O so that OQ approaches u. u is therefore one tangent to the curve through O. Similarly if p' approaches v', Q approaches O as OQ approaches v. So that v is another tangent to the curve at O. The curve has two branches which intersect at O.

Such a point O is called a double point on the curve and every line through O is considered as meeting the curve in two coincident points at O. The only case of a double point which we have met with hitherto is that of the degenerate conic or line-pair, where the intersection of the two lines is a double point.

Observe that a ray through the double point O meets the cubic again at one point only, as it should, to wit, at the point where it is met by the corresponding ray of the pencil of second order. There are two exceptions, however, namely the rays u, v. These meet the curve in three coincident points at O and are known as the **proper tangents** to the curve at O.

It may be shown that, in order that a cubic may have a double point, a certain condition must be satisfied. Hence the cubic of the present Article is not of the most general type.

195. Construction of directions of the asymptotes of the cubic. The points where the cubic meets the line at infinity i^∞ may be constructed as follows. Draw a tangent a to s (Fig. 64) from O. Let a' be the tangent corresponding to the ray a through O. Let c' be the tangent parallel to a, c its corresponding ray. Let b, b' be any other pair of corresponding lines. Let a, b, c meet i^∞ at A^∞, B^∞, C^∞ and a', b', c' meet a at A', B', C'^∞ ($C'^\infty = A^\infty$). If p, p' be any pair of corresponding lines (not shown in Fig. 64) meeting i^∞, a at P^∞, P' respectively the ranges $[P^\infty]$, $[P']$ are projective, and $A^\infty B^\infty C^\infty$; $A'B'C'^\infty$ are two corresponding triads. The parallel $P^\infty P'$ through P' to p therefore envelops a parabola which touches a and i^∞ at A', C^∞ respectively, these being the correspondents to $ai^\infty (= C'^\infty = A^\infty)$. This parabola also touches $B'B^\infty$, *i.e.* the parallel through B' to b. We are therefore given one tangent $B'B^\infty$, another tangent a and its point of contact A', and the direction of the axis parallel to c. The parabola can then be drawn by Brianchon's Theorem. The three common tangents to the parabola and to s, other than a, then give the directions of the three asymptotes of the cubic. For let t' be any one of these common tangents, meeting a at T', and if t be the corresponding ray, meeting the line at infinity at T^∞, $T'T^\infty$ is a tangent to the parabola and therefore in the same straight line as t'. Thus tt' is T^∞, a point at infinity on the cubic.

196. The product of two homographic ranges of the first and of the second order respectively is a curve of the third class with a double tangent. If $[P]$, $[P']^2$ are homographic ranges on a line x and a conic s respectively, and any point U is taken in the plane, then if $p = UP$, the pencil $[p]$ and the range $[P']^2$

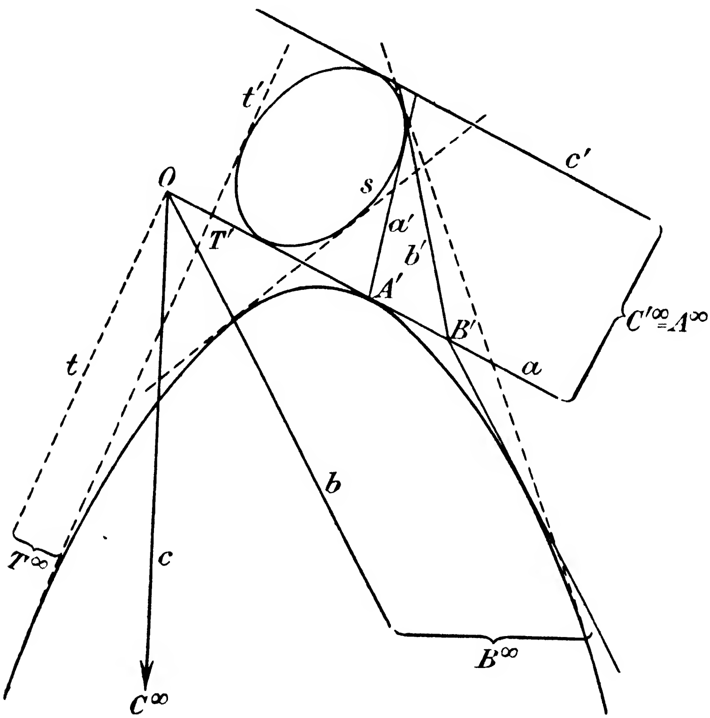


FIG. 64.

are homographic. They have three incident pairs which correspond to those cases where U, P, P' are collinear, *i.e.* where UP is tangent to the envelope of PP'. The envelope is thus of the third class since from any point three tangents can be drawn to it.

Let the two points where x meets s be T', V'. Let the corresponding points on x be T, V. Then x occurs twice as a tangent to the envelope. It is therefore a double tangent and touches the envelope at T, V.

197. Degenerate cases of the above. If the two homographic pencils of Art. 193 have one of the tangents from O to s as a self-corresponding ray, the whole of this ray forms part of the locus. The locus of the third degree breaks up therefore into a straight line and a conic. If the two tangents from O are self-corresponding rays, the whole of each of these rays is part of the locus and the cubic breaks up into three straight lines, namely the two tangents from O and a third tangent to s (Art. 190).

In like manner if the ranges of Art. 196 have a self-corresponding point, that point is an isolated part of the envelope. The envelope

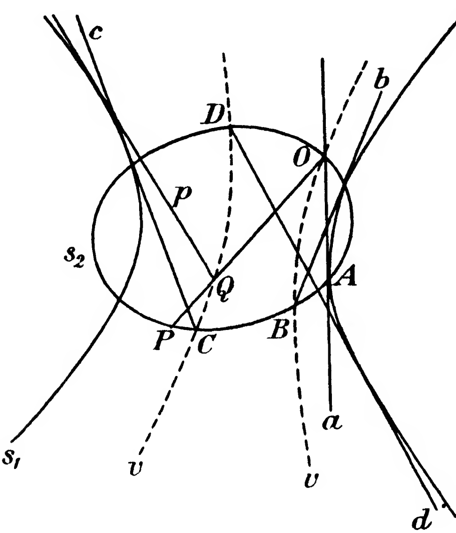


FIG. 65.

breaks up into this point and a curve of the second class, *i.e.* a conic. If the ranges have two self-corresponding points, the envelope breaks up into three points, one of which is on the base of the range of second order by Art. 190.

198. Two homographic unlike forms of the second order have four pairs of corresponding elements incident; and if they have more than four, they are altogether incident. Let $[p]^2$, $[P]^2$ (Fig. 65) be a homographic pencil and range respectively, of the second order, whose bases are s_1, s_2, respectively. On s_2 take a point O. Then the pencils $O[P]$, $[p]^2$ are homographic. Let Q be the meet of p, OP. The locus of Q is a cubic of which O

is a double point. But a conic and a cubic are known from analytical considerations to have six intersections. Hence the locus of Q meets s_2 at six points. Of these O counts as two, since O is a double point on the cubic, and the tangents at O to the cubic are, in general, different from the tangent at O to s_2. There are accordingly four others A, B, C, D. Corresponding to each such intersection we have a point P on its corresponding line (since P and Q are then coincident).

Or we may proceed as follows. Let A be one intersection of s_2 with the locus of Q (it being assumed that at least one such intersection, real or imaginary, exists). Through such a point A the corresponding line a passes. Take for O the point where a meets s_2 again. The pencils $O[P]$, $[p]^2$ have now a for a self-corresponding ray. The locus of Q now reduces to a conic v through O (a being irrelevant). This conic v (shown by the dotted line in Fig. 65) cuts s_2 at three other points B, C, D, which are incident with their corresponding lines. O is not incident with its corresponding line, unless v touches s_2 at O; for if p is the second tangent to s_1 from O, then Q coincides with O and P is the meet of s_2 with the tangent at O to v. P is then distinct from O unless the last-named tangent is also tangent to s_2 at O. But, if v, s_2 touch at O, they have only two other points of intersection and there are still only four points on their corresponding lines.

Hence if there be a fifth point E through which passes its corresponding line e, the conics v, s_2 have five points in common and coincide entirely. Thus every point lies on its corresponding line and the two given forms are incident.

In such a case the bases s_1, s_2 have double contact. For if, as has been proved, every point P lies in its corresponding line p, let r be the second tangent from P to s_1, meeting s_2 again at R. Then either R or P corresponds to r, and since P does not correspond to r, R must do so. If now P coincide with a common point of s_1, s_2, p and r both coincide with the tangent to s_1 at P. Hence the corresponding points P and R also coincide and PR, that is r, is a tangent to s_2 at P, or s_1, s_2 touch at every common point.

199. Product of cobasal homographic forms of the second order. Let $[P_1]^2$, $[P_2]^2$ be two homographic ranges of the second order on the same conic s. Let A_1, A_2 be a given pair of corresponding points of these ranges and ST the cross-axis (Fig. 66 (a)). Then A_1P_2, A_2P_1 meet at U on the cross-axis. Project S, T into the circular points at infinity Ω, Ω'. s projects into a circle s'

(Fig. 66 (*b*)), U into a point U'^{∞}. Therefore $A_1'P_2'$, $A_2'P_1'$ are parallel, and the arcs $A_1'P_1'$, $A_2'P_2'$ are directly equal. The ranges $[P_1']^2$, $[P_2']^2$ on the circle are thus determined by directly equal flat pencils whose vertex is on the circle. The arc $P_1'P_2'$ subtends a fixed angle at the circumference and therefore at the centre. Hence the chord $P_1'P_2'$ touches a fixed concentric circle t'.

Now two concentric circles touch one another at Ω, Ω', as has been shown in the last paragraph of Art. 149. Projecting back into the original figure, P_1P_2 touches a fixed conic t which touches the original conic at S and T.

Reciprocating this theorem, we see that the product of two homographic pencils of tangents to the same conic s is a conic which

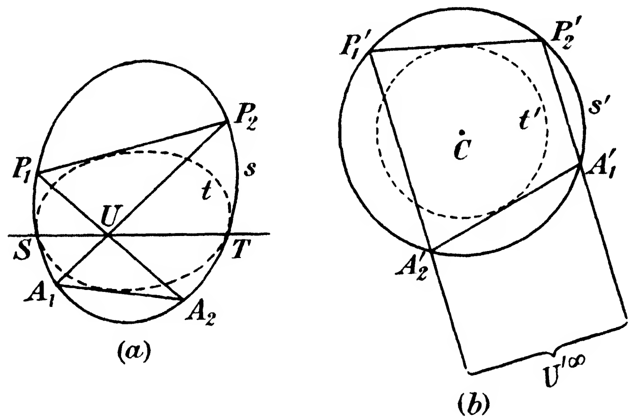


FIG. 66.

has double contact with the original conic, the common tangents meeting at the cross-centre of the pencils.

We have already met a case of the above results in Chapter XI, Arts. 173, 174, when dealing with the base conics of a reciprocal transformation.

In the special case where these homographic forms are in involution, the above envelope and locus degenerate into a point and a straight line respectively, the point appearing as the intersection of the two components of a line-pair and the line as the join of the components of a point-pair. The line-pair in the first case is the pair of tangents from the centre of involution to the conic and the point-pair in the second case is the pair of points at which the axis of involution meets the conic.

The above theorems are of considerable importance and a number of interesting particular deductions flow from them. In particular let there be two ranges on the line at infinity i^∞ defined by the intersections with i^∞ of two directly equal pencils in which corresponding rays make an angle α with one another, and let Q^∞, Q'^∞ be corresponding points of these ranges. Since i^∞ touches every parabola, the tangents from Q^∞, Q'^∞ to a parabola define two homographic pencils of tangents to the parabola. Their product is a conic meeting i^∞ at the two points corresponding to the point of contact of i^∞ with the parabola. Hence we have the theorem : the locus of intersections of two tangents to a parabola which make an angle α (other than right) with one another is a hyperbola whose asymptotes make an angle α with the axis of the parabola.

200. Homographic involutions. We may treat a pair of mates in an involution as a single entity and establish a one-one algebraic correspondence between the pairs of one involution and the pairs of another; such correspondence does not establish any one-one relation between the individual mates, but only between the pairs as a whole. Two involutions correlated in this way will be called homographic.

We will first show how to derive two homographic involution ranges on the same conic s one from the other. Let (P_1, P_1'), (P_2, P_2') (Fig. 67) be two corresponding pairs; let O_1, O_2 be the corresponding involution centres and p_1, p_2 the rays through O_1, O_2 determining the corresponding pairs on s. Then by hypothesis the rays p_1, p_2 are connected by a one-one algebraic correspondence. The pencils $[p_1]$, $[p_2]$ are therefore homographic: if $Q = p_1p_2$ the locus of Q is a conic which meets s at four points A, B, C, D.

When Q is at A, one point of a pair (P_1, P_1') of one involution coincides with one point of the corresponding pair (P_2, P_2') of the other involution, though it should be noted carefully that the pairs as a whole do not in general coincide. Such a point as A will be spoken of as a self-corresponding point of the homographic involutions. Since there are four such points A, B, C, D two homographic involutions of points on the same conic have four self-corresponding points.

But any two involutions, of any type, may always be uniquely correlated with two involution ranges on the same conic, *e.g.* two involution ranges on different straight lines may be projected from a vertex as two concentric involution pencils and then cut by a conic through the vertex; and two involution ranges on different conics may be projected from vertices on the conics into two non-concentric involution flat pencils and then cut by a conic through the two vertices; also two involution pencils of tangents to two conics are correlated with the involution ranges formed by their points of contact.

Hence the above method can ultimately be used to derive geometrically any two homographic involutions one from the other.

Also we have the general theorem that **two cobasal homographic involutions have four self-corresponding elements.**

201. Harmonic envelope and locus. Returning to the case of two homographic involutions on the same conic s (Fig. 67), we have seen (Art. 85) that (P_1, P_1') are harmonically conjugate with respect to (P_2, P_2') if the lines P_1P_1', P_2P_2' are conjugate for s. Let p_3 be the ray through O_1 conjugate to p_2; then $[p_3] \barwedge [p_2] \barwedge [p_1]$. The concentric projective pencils $[p_3]$, $[p_1]$ have, in general, two self-corresponding rays, each of which determines a pair (P_1, P_1') harmonically conjugate with respect to its corresponding pair (P_2, P_2') in the homographic involution.

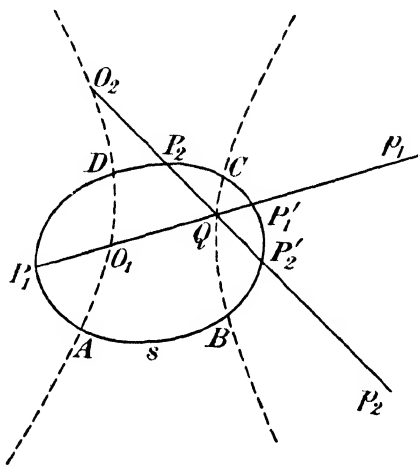


FIG. 67.

Thus any two homographic involutions on the same conic, and therefore **any two cobasal homographic involutions, have two sets of mutually harmonic corresponding pairs.**

An important result immediately follows from this. Let k_1, k_2 be two conics in a plane, O an arbitrary point of the plane, U any intersection of k_1, k_2. A variable ray through O meets k_1, k_2 at pairs of points (P_1, P_1'), (P_2, P_2') respectively, which are clearly corresponding pairs of mates in two homographic involutions on k_1, k_2. Projecting these from U, we have two concentric homographic involution pencils in which (UP_1, UP_1'), (UP_2, UP_2') are corresponding pairs of mates. These two involution pencils have, by the result proved above, two sets of mutually harmonic pairs of mates, for which (UP_1, UP_1') are harmonically conjugate with respect to (UP_2, UP_2'), and therefore (P_1, P_1') are harmonically conjugate with respect to (P_2, P_2'). Thus from any point O in the plane two lines can be drawn, meeting k_1 and k_2 in two mutually harmonic pairs. Hence the envelope of a line which meets two conics in pairs of points which are mutually harmonic, that is, which meets either conic at two points conjugate for the other conic, is a curve to which two tangents can be drawn from any point of the plane, that is, it is a conic. This conic is termed the **harmonic envelope** of the conics k_1, k_2.

Reciprocating the above, we see that the locus of a point such that the tangents from it to a conic k_1 are conjugate for another conic k_2, and conversely, is a conic which is termed the **harmonic locus** of k_1, k_2.

If A be an intersection of k_1, k_2 and the tangent at A to k_1 meet k_2 again at B, B and A are harmonically conjugate with respect to the two coincident points at A, in which the tangent in question meets k_1. Such a tangent is therefore a tangent to the harmonic envelope, which accordingly touches the eight tangents to k_1, k_2 at their four points of intersection. Thus the eight tangents at the intersections of two conics touch a conic, which is the theorem proved otherwise in Art. 150.

Reciprocating, we see that the eight points of contact of the four common tangents to two conics lie on the harmonic locus of the conics.

As a particular case the orthoptic circle is the harmonic locus of its conic and the point-pair Ω, Ω'. It therefore passes through the points of contact of the tangents from Ω, Ω' to the conic.

202. Case where double elements correspond. In general, in two homographic involutions, no homographic relation exists between the individual components of the pairs. In the case where the double elements correspond, however, such a relation can be shown to exist.

We notice first that, since, from Art. 200, the relation between two homographic involutions is determined by a relation between two homographic simple forms, which latter is itself determined by two corresponding triads, two corresponding triads of pairs can be arbitrarily assumed and completely determine the relation between two homographic involutions.

Suppose now that A_1, B_1, the double elements of one involution, and A_2, B_2, the double elements of another involution, correspond. Let the pair (C_1, C_1') correspond to the pair (C_2, C_2'). Then by the property of involutions that any pair are harmonically conjugate with regard to the double elements, we have

$$\{A_1C_1B_1C_1'\} = \{A_2C_2B_2C_2'\},$$

and the sets of four elements $A_1C_1B_1C_1'$, $A_2C_2B_2C_2'$ can be brought into homographic correspondence. Now the homographic correspondence thus defined will transform the involution (P_1, P_1') into a homographic involution (P_3, P_3') cobasal with (P_2, P_2') and homographic with it. But the cobasal homographic involutions (P_3, P_3'), (P_2, P_2') have three self-corresponding pairs, namely the double elements A_2, B_2 and the pair (C_2, C_2'). Therefore they must be identical. Hence this homographic correspondence connects P_1 with P_2 and P_1' with P_2', *i.e.* there is a homographic correspondence between the individual components of the pairs. Similarly a homographic correspondence exists which connects P_1 with P_2' and P_1' with P_2.

203. Product of two homographic involutions of the first order. If we form the product of two homographic involution pencils of vertices O_1, O_2, that is, find the intersections p_1p_2, p_1p_2', $p_1'p_2$, $p_1'p_2'$ where (p_1, p_1'), (p_2, p_2') are two corresponding pairs, these intersections lie on a certain locus.

As in Art. 90 we proceed to find the intersections of this locus with any straight line x. The two involution pencils determine on x two collinear homographic involutions of which (P_1, P_1') (P_2, P_2') are corresponding pairs, where $P_1 = p_1x$, etc. If either of the points P_1, P_1' coincide with either of the points P_2, P_2', the point of coincidence lies on a ray of each of two corresponding pairs of the given homographic involution pencils, that is, it lies on their product.

But, by Art. 200, there are four such self-corresponding points of the collinear homographic involution ranges (P_1, P_1') (P_2, P_2'). The locus therefore meets any straight line in four points, that is, it is a curve of the fourth degree.

Also the vertices O_1, O_2 are double points on this curve. For let (u_1, u_1') be the pair of the pencil vertex O'_1 corresponding to the pair of the pencil vertex O_2 of which O_2O_1 is a component. As the ray p_2 approaches O_2O_1, (p_1, p_1') approach (u_1, u_1') and two points on the locus coincide at O_1, moving ultimately along u_1, u_1'. O_1 is thus a double point, (u_1, u_1') being the proper tangents at O_1. In like manner O_2 is a double point and, if (v_2, v_2') is the pair of the pencil through O_2 corresponding to the pair of the pencil through O_1 of which O_1O_2 is a component, then v_2, v_2' are the proper tangents at O_2.

If O_1O_2 happens to be a self-corresponding ray of the pencils, the locus breaks up into O_1O_2 and a cubic curve passing through O_1, O_2. In this case, however, O_1, O_2 are not double points on the cubic, for the two mates to O_1O_2 are now the only tangents at O_1, O_2.

Also if it so happen that the correspondence between the two involutions is of such a nature that individual mates can be brought into one-one correspondence (Art. 202), the locus of the fourth degree breaks up into two conics, these being the products of the two pairs of homographic pencils formed by the individual mates.

Reciprocating the above theorems or proceeding directly in a similar manner, we have the result that the product of two homographic involution ranges of the first order is a curve of the fourth class, to which the bases of the given involutions are double tangents. If the ranges have a self-corresponding point this envelope breaks up into a point and a curve of the third class. If the individual mates can themselves be homographically correlated, it breaks up into two conics.

204. Involution homographic with a simple form. We can extend this method and define in an analogous manner homography between an involution and a simple form. In order to establish the relations between these, we proceed as in Art. 200, and consider a range on a conic s with a homographic involution range on the same conic. The range may be defined by a pencil $[p_1]$ through a vertex O_1 on s and the involution by a homographic pencil $[p_2]$ through a vertex O_2 not on the conic. We obtain the figure if in Fig. 67 we take O_1 on the conic. All points P_1' then coincide with O_1, so that this is really a special case of the last. The product t of the pencils $[p_1]$, $[p_2]$ now cuts s at O_1 and at *three* other points, and it is easy to show, as in Arts. 191, 198, that O_1 is not a self-corresponding point unless t, s touch at O_1. Hence in a homography between an involution and a simple form on the same conic there are, in general, three self-corresponding points; and the result can be extended to cobasal involutions and forms of any type as in Art. 200.

Proceeding as in Art. 203 we can show that the product of an involution pencil of vertex O_1 and a homographic simple pencil of vertex O_2 is a cubic having O_1 for a double point and passing through O_2. If O_1O_2 be a self-corresponding element the locus breaks up into the line O_1O_2 and a conic through O_1 but not through O_2.

Similarly the product of an involution range on a line u_1 and a simple range on a line u_2 is a curve of the third class having u_1 for a double tangent and u_2 for an ordinary tangent. If u_1u_2 be a self-corresponding point the envelope breaks up into a point and a conic.

205. The product of two homographic pencils of the second order is a curve of the fourth degree. Consider two homographic pencils of tangents about two conics s_1 and s_2. Let u be

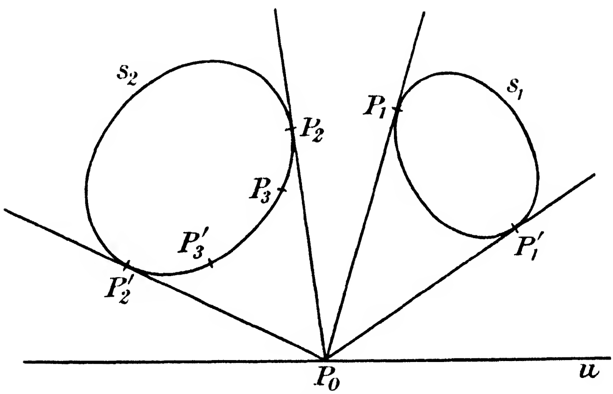


FIG. 68.

any straight line in the plane. Take any point P_0 on u (Fig. 68) and draw from P_0 pairs of tangents to s_1 and s_2 touching these conics at P_1, P_1' and P_2, P_2' respectively. Then the involutions (P_1, P_1') (P_2, P_2') are homographic. Let P_3, P_3' be the points of contact of the tangents to s_2 which correspond in the given homography to the tangents at P_1, P_1' to s_1. Then, owing to this homography, the pairs (P_3, P_3') form an involution homographic with that formed by the pairs (P_1, P_1'), and therefore with the one formed by the pairs (P_2, P_2'). Now there are four self-corresponding elements of the homographic cobasal involutions (P_2, P_2') (P_3, P_3'). To each of these self-corresponding points corresponds a point P_0 such that through P_0 pass two tangents belonging to corresponding pairs of mates in the original involution pencils. Conversely to every such point P_0 corresponds a self-corresponding point of the involutions (P_2, P_2') (P_3, P_3'). But the points P_0 through which pass corresponding tangents of the original pencils lie on the product of the pencils. Any straight line u therefore meets such a product in four points. Hence the locus is a curve of the fourth degree.

If one of the common tangents is self-corresponding it is part of the locus. The latter then breaks up into this line and a cubic. If a second common tangent is self-corresponding the locus breaks up into two straight lines and a conic. The case where three common tangents are self-corresponding has already been discussed in Art. 191. The locus then breaks up into the four common tangents.

Reciprocating the above we see that the product of two homographic ranges of the second order is an envelope of the fourth class. The student will have no difficulty in tracing the degenerate cases when one, two, or three points are self-corresponding.

EXAMPLES XIIA

1. If $[p]$, $[p']^2$ be two homographic pencils of the first and second orders respectively having a self-corresponding ray a which touches the base s of $[p']^2$ at A; and if u be any straight line in the plane, and $up=P$, $ap'=P'$: prove that PP' envelops a conic which touches s at A.

2. If two given homographic pencils of the first and second orders respectively have a self-corresponding ray, show how to construct the two intersections of their product with any straight line.

3. If two given homographic ranges of the first and second orders respectively have a self-corresponding point, show how to draw the two tangents to their product from any point.

4. From a point O a ray OP is drawn to meet a fixed straight line l at P. If O' be the point of contact of a tangent from O to a fixed conic and $O'P$ meet the conic again at P', prove that the locus of the intersection of OP and the tangent at P' is a conic.

5. P, Q are two points on a tangent to a conic s. From P, Q tangents p, q are drawn to s, meeting at R. If PQ be of constant length, find the locus of R.

6. Through a fixed point O a ray is drawn to meet a given circle at P. Find the envelope of a straight line through P which makes a constant angle with OP.

7. Show that if two conics have double contact, any tangent to either determines on the other ranges homographic with each other and with the range described by the point of contact.

8. Two conics touch at A and B. A chord PQ of one conic slides on the other. Show that the cross-ratio of the four points A, B, P, Q is constant.

9. P is a point on a conic s; from P a tangent is drawn to a conic t which has double contact with s to meet s again at P_1; from P_1 another tangent is drawn to t to meet s again at P_2: and so on. After n operations we reach

a point P_n by a chain of tangents. If the chain of tangents slide round t, prove that the ranges $[P]^2$, $[P_1]^2$, $[P_2]^2, \ldots [P_n]^2$ are all projective and have common self-corresponding points. Prove also that if for one position of P (other than a point of contact of s, t) P_n coincides with P, it will do so for all positions of P.

Deduce that if a polygon of n sides exist which can be inscribed in a conic s and circumscribed to a conic t having double contact with s, an infinite number of such polygons exist.

10. A straight line meets a conic at A, B. On AB points P, Q are taken so that the cross-ratio $\{ABPQ\}$ is constant. From P and Q tangents are drawn to the conic meeting at R. Show that R lies on either of two fixed conics having double contact with the original conic at A and B.

[Project A, B into the circular points.]

11. Prove that a variable circle which cuts two fixed circles at right angles determines on these circles two homographic involutions.

12. s_1, s_2 are two conics, u a fixed tangent to s_1. From a point Q_0 of u tangents Q_0Q_1, Q_0Q_2 are drawn to s_1, s_2. O_1, O_2 are fixed points on s_1, s_2 respectively: O_1Q_1, O_2Q_2 meet at Q. Show that the locus of Q is a cubic having O_2 for a double point and construct the proper tangents to the cubic at O_2.

13. If in two homographic involution pencils of the first order the join of the vertices O, O' is a double ray of each pencil and self-corresponding, prove that the remainder of the product is a conic with regard to which O, O' are conjugate points.

[For the other double ray through O meets its corresponding pair in the points of contact of that pair and so is the polar of O'.]

14. A cubic curve is given as the product of a pencil of the first order and a homographic pencil of the second order. Given the vertex of the pencil of the first order, the base of the pencil of the second order and three points on the cubic, show how to construct the cubic and prove that there are eight solutions.

15. Prove the following construction for the tangent to a cubic with a double point O at any point P on it, given any other point U on the cubic and the points of contact A, B of the tangents from U to the cubic.

Let the conic passing through O and touching UA, UB at A, B meet OP at Q. Let the tangent at Q to this conic meet UA, UB at C and D. Then the tangent at P to the conic through O, U, C, D, P is also the tangent to the cubic.

16. If S be a double point on a cubic, O another fixed point on the cubic, OPQ a ray through O cutting the cubic again at P, Q, show that SP, SQ are mates in an involution.

Hence show that if O, A, B are collinear, and if O, C, D are collinear, the locus of the points of contact of tangents from O to all cubics having a common double point and passing through O, A, B, C, D consists of two straight lines through the double point.

17. Show that the converse of Art. 203 is not, in general, true, that is, every quartic with two double points O_1, O_2 cannot be obtained as the product of homographic involution pencils with vertices O_1, O_2.

18. Prove that the product of two pencils of the first and second orders respectively, which are homographic and have two self-corresponding rays, is a straight line touching the base of the pencil of the second order.

19. Show that, through each of the vertices O_1, O_2 of two homographic involution pencils, four tangents can be drawn to the quartic curve which is the product of these involutions.

20. If O_1, O_2 are the vertices of two homographic involution pencils, prove that, if (O_1A, O_1B), (O_2A, O_2B) are the double rays of these involutions, and a conic s be drawn through O_1, O_2, A, B, then the given involutions are obtained by projecting from O_1, O_2 two homographic involutions on s with a common centre U, and that the intersections of s (other than O_1, O_2) with the quartic which is the product of the involution pencils lie on the self-corresponding rays of the homographic pencils through U which determine the involutions on s.

21. Prove that any circle through the vertices of two homographic rectangular involutions meets the quartic which is the product of these involutions at the four corners of a rectangle.

22. If, in two homographic involution pencils, vertices O_1, O_2, a double ray O_1A corresponds to a double ray O_2A, prove that A is a double point on the quartic which is the product of the involutions, and that the proper tangents at A are harmonically conjugate with regard to AO_1, AO_2.

23. Show that the hyperbola which is the locus of intersections of tangents to a parabola making a constant angle α with each other has the same focus and directrix as the original parabola.

[Show that the points of contact of the tangents from Ω, Ω' to the parabola lie on the hyperbola.]

24. Prove that the product of two homographic involutions of tangents to the same conic is, in general, a curve of the fourth degree, together with four straight lines; but that, if the homography between the two given involutions is itself involutory, the product consists of four straight lines and a conic counted twice.

25. If u and v are two tangents to a conic k whose axes are x, y, and are such that u makes with x, y angles equal to those which v makes with y, x, but u, v are not at right angles, prove that the locus of uv is a rectangular hyperbola whose vertices are the foci of k.

EXAMPLES XIIb

[The axes of co-ordinates are rectangular.]

1. O is a point on a circle of radius 2 inches, of which C is the centre; U is a point on the tangent to the circle at O, distant 3 inches from O. V is a point on the internal bisector of the angle COU, distant 5 inches from O.

P is a variable point of the circle and UP meets OC at Q, VQ meets OP at R. Trace the locus of R, and find where it is met by the diameter of the circle perpendicular to OC. Show that this locus has a cusp at O.

2. The points A_1 (0, 2) and A_2 (3, 0) are corresponding points of two homographic ranges on the ellipse

$$\frac{x^2}{9}+\frac{y^2}{4}=1,$$

of which $x+2y=6$ is the cross-axis.

Construct by tangents the product of these ranges.

3. U is a point on a circle of radius 2 inches and centre C, and O is a point outside the circle, at a distance of 3 inches from C, on a diameter of the circle

perpendicular to the diameter through U. O is the centre of an involution on the circle, and a ray through U perpendicular to the join of two mates in the above involution meets the circle at P.

Find the real self-corresponding point of the range $[P]^2$ and the involution. What are the imaginary self-corresponding points ?

4. An involution and a range on a straight line are homographic, the homography being defined as follows :

The double point $A(x=2)$ of the involution corresponds to $P(x=3\cdot5)$; the double point $B(x=-2)$ corresponds to $Q(x=-7/6)$; the pair of mates $C_1(x=1)$ and $C_2(x=4)$ correspond to $R(x=7/3)$.

Construct the self-corresponding points of the involution and the range, and prove that your construction must lead to the solution of the cubic equation

$$x^3-2x^2-3x=0.$$

5. O is a point on a circle of radius 1·5 inches, and OP, OQ are two rays through O inclined at a constant angle of 30°. If the tangent to the circle at Q meet OP at R, construct the locus of R by points.

Find independently the points where this locus meets the lines perpendicular to the diameter through O and distant 3 inches from O.

Find also the directions of the real points at infinity on the locus.

6. OX, OY are two lines inclined at 45° to one another ; A, B, C are points in order on OX, such that $OA=AB=BC=1$ inch ; P, Q are points on OY such that $OP=1\cdot5$ inches, $OQ=2$ inches ; AP, BQ meet at U and UC meets OY at R.

A range on OY is homographic with an involution on OX of which A, B are double points, and P, Q, R correspond respectively to A, B and the pair of mates of which C is one.

Construct by tangents the product of the range and the involution and find the points of contact of the double tangent to the envelope.

7. Two variable rays VP, VQ, at right angles to one another, are drawn through the point V (0, 2), to meet the circles $(x-3)^2+y^2=4$ and $(x+2)^2+y^2=1$ at P and Q respectively. If A is the point (1, 0) on the first circle, and B is the point $(-3, 0)$ on the second circle, and AP, BQ meet at R, trace the locus of R, and find its intersections with the straight line $y=x+1$.

8. U is the point (0, 4), P is a point on the line $y=0$, V is the point (3, 3) on the circle $(x-3)^2+(y-2)^2=1$. If VP meet the circle again at Q, and the tangent at Q meet UP at R, prove that the locus of R is a cubic, and draw it.

CHAPTER XIII

SYSTEMS OF CONICS

206. Ranges and pencils of conics. A set of conics passing through four fixed points A, B, C, D are said to form a **pencil of conics.**

Through any fifth point E of the plane there passes one conic of the pencil and one only, since five points determine a conic.

The four points A, B, C, D may be referred to as the **base points** of the pencil, and as forming its **base quadrangle.**

The conics touching four fixed lines a, b, c, d are said to form a **range of conics.**

There exists one conic of the range, and one only, which touches any given line e of the plane.

The four lines a, b, c, d will be said to be the **base lines** of the range and to form its **base quadrilateral.**

Special cases of pencils and ranges of conics are obtained when two or more of the four points A, B, C, D, or of the four lines a, b, c, d, are coincident.

Thus the conics which touch a line a at one of its points A and also pass through two other fixed points B and C, form a pencil of conics. Again the conics which touch a at A and touch two other lines b, c, form a range of conics.

A particularly important case is when two pairs of points, or two pairs of lines coincide. Thus the conics which touch a line a at A, and a line b at B, form a pencil. But they likewise form a range. Thus such a system of conics, which touch at A and B, possesses the properties both of a pencil and of a range of conics. By Art. 149 they can be projected into concentric circles, and they possess all the projective properties of such circles.

Again, we may make three points A, B, C coincide. We then have conics, having three-point contact at A with a given conic s and passing through a fixed point D. These form a pencil of conics. Or, if we make three tangents a, b, c coincide, A being their point of contact, we have conics having three-line contact (which, by Art. 47 is the same as three-point contact) at A with a given conic s, and touching a fixed line d. These form a range of conics.

Finally, by making all four points A, B, C, D, or all four lines a, b, c, d, coincide, we obtain the set of conics having four-point contact (or four-line contact) with a given conic s at A. Such a system of conics is both a pencil and a range, like a system of conics having double contact (of which it is a particular case).

It is clear that, of the conics of a pencil, three are line-pairs, namely the three pairs of opposite sides of the base quadrangle.

Similarly, of the conics of a range, three are point-pairs, namely the three pairs of opposite vertices of the base quadrilateral.

Since by Art. 50 the diagonal triangle of a quadrangle inscribed in a conic is self-polar for the conic, it follows that the diagonal triangle of the quadrangle $ABCD$ which is inscribed in all the conics of the pencil is self-polar with regard to every conic of the pencil. Clearly the vertices of this triangle are the centres (*i.e.* the intersections of the components) of the three line-pairs of the pencil.

In like manner the diagonal triangle of the quadrilateral $abcd$ is self-polar with regard to all the conics of the range defined by a, b, c, d. Its three sides are the lines joining the components of the three point-pairs of the range.

The above no longer holds good if two or more of the base elements coincide. In such cases it will be found that there is no proper diagonal triangle, except when the conics touch at A and B, when the diagonal triangle is indeterminate, being formed by the meet of the common tangents and by any two points harmonically conjugate with respect to A and B.

Example

Prove that, if all the base points A, B, C, D of a pencil of conics coincide at O, in such a way that AB, CD coincide with a determinate line OX, and AC, BD coincide with a determinate line OY, the pencil of conics reduces to an involution pencil of the first order, of which OX, OY are the double rays.

207. Involutions determined by pencil or range of conics with a straight line or point respectively. Consider any straight line u not passing through a base point of the pencil. Let P be any point of u. The conic of the pencil through P meets u again at one point P', which is therefore uniquely determined if P be given. Conversely if P' be given P is known. Also, since P and P' determine the same conic of the pencil, when P

is taken at P', P' is at P. The ranges $[P]$, $[P']$ on u are therefore connected by a one-one correspondence in which the elements correspond doubly. Hence they form an involution upon u.

The double points S, T of this involution are the points of contact of the conics of the pencil which touch u. This enables us to solve the problem: to draw a conic through four given points A, B, C, D and touching a given straight line u. We see that this problem has in general two solutions which are real only if the involution determined upon u by the pencil of conics is hyperbolic.

Three of the conics of the pencil degenerate into the line-pairs formed by opposite sides of the quadrangle $ABCD$. They are $(AB; CD)$, $(AC; DB)$, $(AD; BC)$. We thus obtain the theorem:

The three pairs of opposite sides of a complete quadrangle meet any straight line (not passing through a vertex of the quadrangle) in three pairs of points of an involution (*cf.* Art. 95, Ex. 6).

Proceeding on similar lines, or reciprocating the above theorem, we obtain the result:

If (p, p') be the tangents from a point U not lying on a base line of the range to any conic of a range, the rays (p, p') form an involution, of which the double rays s, t are the two tangents at U to the conics of the range which pass through U.

The problem, to draw a conic to touch four given lines and to pass through a given point, has therefore in general two solutions.

Three of the conics of the range degenerate into the point-pairs formed by opposite vertices of the quadrilateral $abcd$. They are $(ab; cd)$, $(ac; db)$, $(ad; bc)$. The tangents from U to these point-pairs are the joins of U to the points of the pair. We have then:

The lines joining any general point to the three pairs of opposite vertices of a complete quadrilateral form three pairs of mates of an involution pencil.

From the property of the present Article follows at once the theorem that **the orthoptic circles of a range of conics are coaxal.**

For consider two such orthoptic circles, intersecting at G, H. The involution of tangents to the system from G has two pairs of rectangular rays and is therefore rectangular. Thus G lies on every orthoptic circle of the system. Similarly for H.

EXAMPLES

1. Prove that, in general, the condition that two given points S, T are conjugate with respect to a conic of a pencil determines this conic uniquely. Discuss the case of exception.

[The intersections of the required conic with ST are the common mates of the involution determined by the pencil on ST and the involution of which S, T are double points. These involutions may coincide (*see* Art. 210).]

2. Prove that, in general, the condition that two given lines s, t are conjugate with respect to a conic of a range determines this conic uniquely.
Discuss the case of exception.

3. The conics k, k' intersect at four points A, B, C, D; a line l meets AB, CD at X, Y respectively, and meets k at P, Q. If P, X be conjugate with respect to k', prove that Q, Y are also conjugate with respect to k'.

4. Four given points A, B, C, D lie on a line l; and $GHJK$ is any quadrangle such that GH, JK meet at A, and GJ, GK, HJ pass through B, C, D, respectively. Prove that the intersection of HK with l is a fixed point, independent of the particular quadrangle $GHJK$.
If E is a fifth given point of l, show that the second intersection of l with the conic $EGHJK$ is also a fixed point independent of the particular quadrangle $GHJK$; and that this conic touches l if, and only if, (E, A) are harmonic with respect to (C, D).

5. The conics of a pencil touch a fixed line l at A and pass through two fixed points B, C. Prove that, if a conic of this pencil meets any other fixed conic k touching l at A (but not passing through B, C) at two other points P, Q, then PQ passes through a fixed point of BC.

6. Show that a pencil of conics which either (i) have three-point contact with a fixed conic k at a given point A and pass through a fixed point B, or (ii) have four-point contact with k at A, determine an involution on a fixed conic s having simple contact with k at A.

7. Show that the circles on the three diagonals of a quadrilateral as diameters have a common radical axis.
[For these circles are orthoptic circles of the point-pairs of the range.]

208. Homographic ranges and pencils of conics. Two pencils of conics $[k_1]$, $[k_2]$ will be said to be **homographic** if, given one conic k_1 of one pencil, a conic k_2 is uniquely determined by any process which can be expressed by means of an algebraic correspondence, and, conversely, when k_2 is given, k_1 is uniquely known.

Thus, if the pencil $[k_1]$ pass through the fixed points A_1, B_1, C_1, D_1 and the pencil $[k_2]$ pass through the fixed points A_2, B_2, C_2, D_2, then, if $[u_1]$, $[u_2]$ are homographic flat pencils through A_1, A_2 respectively, k_1 is entirely determined if it touches u_1, and k_2 is entirely determined if it touches u_2. But u_1, u_2 are uniquely related, so that, if k_1 is given, the tangent u_1 to k_1 at A_1 is known, hence u_2, and therefore k_2, is determined. Thus the pencils $[k_1]$, $[k_2]$ are homographic. Conversely, if two pencils $[k_1]$, $[k_2]$ are homographic, the tangents u_1, u_2 at base points A_1, A_2 are homographic.

It follows that the homographic correspondence between two pencils of conics is entirely determined if three pairs of corresponding conics (k_1', k_2'); (k_1'', k_2''); (k_1''', k_2''') in the pencils, are given. For then in the homographic pencils $[u_1]$, $[u_2]$ we have three pairs of corresponding rays (u_1', u_2'); (u_1'', u_2''); (u_1''', u_2'''), and these determine the homographic relation between $[u_1]$, $[u_2]$, and hence between $[k_1]$, $[k_2]$.

Again, consider a fixed conic k through four points A, B, C, D, and the pencils of conics $[k_1]$ through A_1, B, C, D and $[k_2]$ through A_2, B, C, D, where A_1, A_2 are fixed points different from A. If the conics k_1, k_2 are so related that they both pass through the same variable point P of k, $[k_1]$ and $[k_2]$ are homographic. For, if k_1 is given, its fourth intersection P with k is unique and this fixes k_2, and conversely.

In the same way as for pencils of conics, so ranges of conics which have a one-one correspondence between them will be said to be homographic.

In like manner a range of conics may be homographic with a pencil of conics; for example, such a one-one correspondence is given by associating the conic through four given points A, B, C, D which touches a variable line u through A with the conic touching four fixed lines a, b, c, d which touches the same line u—or which touches the line v corresponding to u in a pencil of first or second order homographic with $[u]$.

Again either a range, or a pencil of conics, may be homographic with any other type of one-dimensional geometric form, or with an involution.

209. The quartic and cubic derived from pencils of conics. If $[k_1[$, $[k_2]$ be two homographic pencils of conics, these will determine on a general straight line u two homographic involutions, which have four self-corresponding points (Art. 200). These self-corresponding points are points of the product of the two pencils. This product therefore meets any straight line at four points, and so is a quartic curve. It is clear that the four base points of each pencil lie on this quartic.

It will now be shown that any quartic q may be derived in this manner. Take any four points A_1, B_1, C_1, D_1 on q and through these describe any conic k_1'. Using the proposition that two curves of degrees m and n intersect in mn points, k_1' will intersect q at four points P, Q, R, S, besides A_1, B_1, C_1, D_1. Take now a second conic k_2' passing through P, Q, R, S; k_2' will meet q again at four points A_2, B_2, C_2, D_2. Let now T, U be any two other points on q. Let k_1'', k_1''' be the conics of the pencil defined by the base quadrangle $A_1B_1C_1D_1$ which pass through T, U respectively, and let

k_2'', k_2''' be the conics of the pencil defined by the base quadrangle $A_2B_2C_2D_2$ which pass through T, U respectively. The triads (k_1', k_1'', k_1''') and (k_2', k_2'', k_2''') define a homography between these two pencils of conics, which we now denote by $[k_1]$, $[k_2]$. The product of $[k_1]$, $[k_2]$ is a quartic, which necessarily passes through the eight base points A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, through the intersections P, Q, R, S of the corresponding conics k_1', k_2', through the intersection T of the corresponding conics k_1'', k_2'' and through the intersection U of the corresponding conics k_1''', k_2'''. These give fourteen points on the quartic locus. But the general equation of a quartic contains fifteen coefficients, whose fourteen ratios determine the quartic. Thus through fourteen points only one quartic can in general be drawn. As, however, two quartics intersect in sixteen points, it follows that there must exist sets of fourteen points on any quartic through which more than one quartic can be drawn. We notice, however, that, for this case of exception to arise, there must be some relation between the fourteen points. But, in this case, we can obviously vary arbitrarily one of the points, say U, without affecting any of the thirteen others, and therefore arrange so that the case of exception shall not arise. Thus, in general, the quartic product of $[k_1]$, $[k_2]$ is identical with q.

It will be noticed that this does not necessitate that the quartic considered should have double points, as in the case of a quartic obtained as the product of homographic involution pencils of the first order.

Note also that the twenty points obtained from the eight base points of two pencils of conics and the twelve intersections of any three pairs, each consisting of one conic from each pencil, lie on a quartic.

If now we consider a cubic c and proceed in a similar manner, taking four arbitrary points A, B, C, D on c and a conic k' through them, k' will meet the cubic at two other points P, Q. Join $PQ=u'$, meeting the cubic again at Z. If R, S be two other points on the cubic, and we denote by k'', k''' the conics $ABCDR$, $ABCDS$ respectively, and by u'', u''' the lines ZR, ZS respectively, then if a homographic correspondence is established between the pencil of conics $[k]$ through A, B, C, D and the pencil of rays $[u]$ through Z, in which u', u'', u''' correspond to k', k'', k''' respectively, the product clearly passes through the nine points Z, A, B, C, D, P, Q, R, S. Now nine points, in general, determine a cubic; here again, since two cubics meet at nine points, cases of exception may arise, which may be removed by varying R or S. Thus, generally, the product of $[k]$ and $[u]$ is the original cubic c.

We obtain at once the following important theorem on the cubic: if, through four given points A, B, A', B' of a cubic a pencil of conics be drawn, and any conic of the pencil meets the cubic again at points P, Q, the lines PQ pass through a fixed point Z of the cubic.

Take the line-pairs $(AB, A'B')$ and (AA', BB') as two of the conics. Let them meet the cubic again at C, C' and at X, Y respectively. Then CC', XY meet at Z on the cubic. Thus: **if any two straight lines meet a cubic at** A, B, C; A', B', C' **respectively, and if** AA', BB', CC' **meet the cubic again at** X, Y, Z, **then** X, Y, Z **are collinear.**

As a particular case let the two straight lines coincide. Then the tangents at the three points where a line meets a cubic intersect the cubic again at three collinear points. If the line is the line at infinity, we have the result that the intersections of a cubic with its asymptotes are collinear.

It should be noticed that the point Z can be arbitrarily selected on the cubic, instead of the four points A, B, C, D. For we can draw any line through Z meeting the cubic again at P, Q and through P, Q a conic k' meeting the cubic at A, B, C, D, and then proceed as before.

In a similar manner we can derive the general envelope of the fourth class from two homographic ranges of conics and that of the third class from a range of conics and a homographic range of the first order.

Examples

1. Show that a quartic determined by two homographic pencils of conics or a cubic determined by a pencil of conics and a homographic pencil of first order cannot have double points unless one of the pencils of conics reduces to an involution pencil of the first order.

2. Show that through any 13 given points on a quartic a pencil of quartics can in general be drawn, which meets the original quartic at three other fixed points, and that, through any other given point of the plane there passes just one quartic of the pencil, in general.

3. If k_1, k_2, s_1, s_2 be four conics, prove that through the 16 intersections of type SK pass an infinite number of quartics, and that 10 of these 16 points may be arbitrarily assumed.

4. Prove that, if k_1, k_2 be two conics of a pencil through A, B, C, D, and u_1, u_2 two rays of a pencil of the first order vertex O; and if u_1 meets k_1 at P_1, Q_1, and u_2 meets k_2 at P_2, Q_2, an infinite number of cubics can be described through the nine points $A, B, C, D, P_1, P_2, Q_1, Q_2, O$; and that through any general point of the plane there passes just one of these cubics.

210. Points conjugate for a pencil of conics. If two points S, S' are conjugate for two conics k_1, k_2 of a pencil, they are conjugate for every conic of the pencil. For let k_1, k_2 meet SS' at P_1, Q_1; P_2, Q_2 respectively: then, since S, S' are harmonic with respect to P_1, Q_1 and P_2, Q_2, they are the double points of the involution determined on SS' by the pencil (Art. 207). Hence, if any other conic k of the pencil meets SS' at P, Q, these latter are mates in the above involution, and S, S' are harmonic with respect to P, Q and therefore conjugate for k.

Clearly there must be in general one pair of points conjugate for the pencil of conics on every straight line l of the plane; they are the double points of the involution determined by the pencil upon l.

To any point S of the plane corresponds, in general, one point S' and one only. For let s_1, s_2 be the polars of S for k_1, k_2 respectively. Then S' must be a point common to s_1, s_2. In general there is only one such point. The only exception arises when s_1 and s_2 coincide, in which case S has the same polar s for all the conics of the pencil. If this happens, then by joining S to a base point A of the pencil and producing to A', where AA' is harmonically divided by S and s, we obtain a point A' which lies on every conic of the pencil, and so must be another of the base points, B, C or D. Such points S then, which have the same polar with respect to every conic of the pencil, are necessarily at an intersection of opposite sides of the base quadrangle. It follows that, if this quadrangle be not degenerate, the pencil of conics can have only one common self-polar triangle, namely the diagonal triangle of the base quadrangle, as in Art. 206.

If, however, the conics touch at A and pass through B and C, there are only two points S having this property, namely A and the intersection of BC and the tangent at A. If, in addition, B or C, or both, coincide with A, so that the conics have three- or four-point contact at A, then A is the only point which has the same polar for all the conics. But if C coincides with B, as well as D with A, so that the conics have double contact, S may be either the intersection T of the tangents at A and B, or any point of AB. In this case any two points U, V of AB which are harmonically conjugate with respect to A, B are conjugate for the pencil and every triangle TUV is self-polar for the pencil.

Returning now to the general case, we see that the polars s of S for the conics of the pencil $[k]$ form a pencil $[s]$ of the first order with vertex S'. Clearly if k is given, s is uniquely determined. Conversely, if s is given and SA is joined, we find on SA, as before, a point A' which must lie on k. Since by hypothesis S is not now a vertex of the common self-polar triangle, A' does not coincide with any of the base-points. The conic k is now uniquely determined by the five points A, B, C, D, A'. It follows that the polars of S form a pencil of first order homographic with the given pencil of conics.

We find an immediate application of this result in the case of the cubic. If S is any point of a cubic c, we have seen that the cubic is obtainable as the product of a pencil $[p]$ of the first order through S and a homographic pencil $[k]$ of conics. If now s is the polar of S for the conic k, $[s]\barwedge[k]$ by what has been proved above. Thus $[s]\barwedge[p]$ and $sp=R$ describes a conic through S, S'. But if p meets k at P and Q, R is harmonically conjugate to S with respect to P and Q. Now P and Q are points on the cubic. Thus if through a point S of a cubic a chord SPQ be drawn, meeting the cubic again at P, Q, the locus of the point harmonically conjugate to S with respect to P and Q is a conic passing through S. This is known as the **polar conic** of S with respect to the cubic.

Examples

1. Prove that if two conics of a pencil have their axes parallel, all the conics of the pencil have their axes parallel and one of these conics is a circle.

2. A coaxal system of circles being given, show that: (i) Any given straight line is touched by two circles of the system; (ii) The polars of a given point with respect to the circles of the system pass through a fixed point.

3. A pencil of conics has three-point contact with a circle at A (AT being the common tangent) and passes through a point B on the circle. Prove that the axes of the two parabolas of the pencil are parallel to the internal and external bisectors of the angle TAB.

4. Prove that the polar conic of a point O on a cubic touches the cubic at O.

5. Show that from a point O on a cubic four tangents can in general be drawn to the cubic, and that the conic through O and the four points of contact touches the cubic at O.

211. Lines conjugate for a range of conics. Proceeding in a similar manner, or reciprocating the results of the last Article, we can show that:

If two lines s, s' are conjugate for two conics k_1, k_2 of a range, they are conjugate for every conic of the range.

Two such lines pass, in general, through any given point P of the plane.

To any given line s of the plane corresponds in general one line s'

and one only, which is conjugate to s for all the conics of the range, except when s has the same pole for all the conics of the range, in which case s is a diagonal of the base quadrilateral, provided the latter is non-degenerate.

The poles of a given line s with respect to the conics of a range form a range of the first order homographic with the range of conics.

If the base quadrilateral is non-degenerate, its diagonal triangle is the only common self-polar triangle of the range.

If, however, the base lines a and d coincide, the only lines which

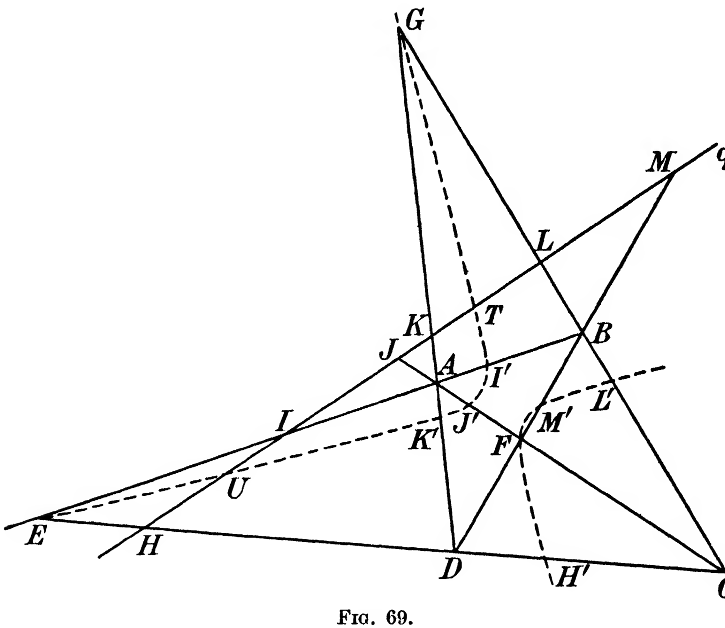


FIG. 69.

have the same pole for the conics of the range are a and the line joining bc to the point of contact of a. If three or four of the base lines coincide with a, a is the only line having the same pole for all the conics. In either of the above cases there is no proper common self-polar triangle.

If the base lines coincide in pairs, d with a and c with b, we have conics having double contact, a case already discussed.

212. The eleven-point conic. We proceed to find the locus of the point S' conjugate to S for a pencil of conics, when S describes a straight line q (Fig. 69). Let s_1, s_2 be the polars of S with respect to two conics k_1, k_2 of the pencil. Then $S' = s_1 s_2$. If Q_1, Q_2 be the poles of q with respect to k_1, k_2, s_1, s_2 describe pencils of the first order with Q_1, Q_2 as vertices and by Art. 52, $[S] \barwedge [s_1] \barwedge [s_2]$. Hence S' describes a conic (Art. 41) passing through Q_1, Q_2.

This conic is known as the **eleven-point conic** of q.

For let EFG (Fig. 69) be the common self-polar triangle of the pencils. Then E is conjugate to the point of q in which q is cut by FG. Therefore E is a point on the locus of S': similarly F, G are points on this locus. Again the two double points T, U of the involution determined by the pencil on q, being conjugate to one another, are on the locus.

Let H, I, J, K, L, M be the points at which q meets the six sides of the quadrangle $ABCD$. Then the harmonic conjugates H', I', J', K', L', M' of H, I, J, K, L, M respectively with regard to the two vertices on the corresponding sides of the quadrangle must lie on the locus. For clearly CD being a chord of all the conics of the pencil (H, H') are conjugate with regard to all such conics.

The locus of S' thus passes through these eleven points.

Since the eleven-point conic of q passes through the two poles Q_1, Q_2 of q with regard to k_1, k_2; and k_1, k_2 are any conics of the pencil, the eleven-point conic passes through the poles of q with regard to all the conics of the pencil.

It is therefore also the locus of the poles of q with regard to the conics of the pencil.

213. The eleven-line conic. Reciprocating the above theorems we obtain the following results.

The envelope of lines conjugate to the rays of a pencil through a point Q with regard to a range of conics touching a, b, c, d is a conic which touches: (1) the three sides of the diagonal triangle of the quadrilateral $abcd$; (2) the two lines through Q conjugate with regard to the range, *i.e.* the two tangents at Q to the two conics of the range through Q; (3) the six harmonic conjugates to the rays joining Q to the vertices of the complete quadrilateral $abcd$, taken with regard to the two sides of the quadrilateral through each vertex.

This conic is also the envelope of the polars of Q with regard to the conics of the range.

EXAMPLES

1. If a pencil of conics circumscribe a rectangle, show that the eleven-point conics are rectangular hyperbolas.

2. Prove that the circle through the feet of the perpendiculars from the vertices of a triangle upon the opposite sides passes also through the middle points of the sides and through the middle points of the three lines joining the orthocentre to the vertices of the triangle.

Show that the centre of this circle bisects the line joining the orthocentre to the circumcentre.

3. Show that the eleven-point conic breaks up into two straight lines if, and only if, q passes through a diagonal point of the base quadrangle, and that, in this case, one straight line is the locus of the poles of q and the other is the locus of the points conjugate to points of q for the pencil.

4. Prove that the eleven-point conics of the lines u through a given point O form a pencil of conics homographic with $[u]$.

214. Geometrical constructions for common self-polar triangle of two conics. If two real conics intersect in four real points A, B, C, D, or else lie entirely outside each other, so that they have four real common tangents a, b, c, d, their common self-polar triangle is at once constructed, being the diagonal triangle of the quadrangle $ABCD$ or of the quadrilateral $abcd$.

If, however, two of the points of intersection, say C and D, are conjugate imaginary, the other two A and B being real, the line CD is a real straight line. The vertex E (Fig. 69) of the self-polar triangle is therefore real and its polar FG with regard to the two conics is also real. But F and G cannot be real: for if F were real, AF would be a real line and its meet C with CD would be a real point, which is against the hypothesis. In this case, then, two vertices of the common self-polar triangle, and also their opposite sides, are imaginary.

If all the points of intersection are imaginary they fall into two conjugate imaginary pairs, say A, B and C, D. Then AB, CD are real lines and their meet E a real point. Also A being conjugate imaginary to B and C conjugate imaginary to D, the line AC is conjugate imaginary to BD by Art. 138, and thus their intersection F is real. Similarly G is real. Hence when all four points of intersection are imaginary, the common self-polar triangle is real.

Proceeding similarly we see that the common self-polar triangle is real when the four common tangents are either all real or all imaginary: it is imaginary when two of the common tangents are real and two imaginary.

Comparing these results with the previous ones we observe that if two conics have only two real intersections they have only two real common tangents; but they may have: (1) four real intersections and four real common tangents, *e.g.* two conics having their four real intersections on the same branch of each; (2) four real intersections and four imaginary common tangents, *e.g.* two conics having real intersections on both branches of one of them; (3) four imaginary intersections and four real common tangents, *e.g.* two ellipses lying entirely outside one another; (4) four imaginary intersections and four imaginary common tangents, *e.g.* one conic lying entirely inside another.

In case (4) the geometrical construction for the diagonal triangle fails entirely. We can then proceed as follows. Take any two lines p and q. Construct their eleven-point conics as in Art. 212. These two eleven-point conics intersect in four points, namely the vertices E, F, G, of the common self-polar triangle and the point conjugate to pq with regard to both conics. The latter point being always real, we get a new proof that one of the vertices of the self-polar triangle is always real.

215. Given two conics of a pencil, to construct any conic of the pencil. Let s, t be two given conics; it is required to construct the conic of the pencil of which s, t are members, which passes through a given point P.

If s, t intersect at four real points, the conic is immediately constructed by Pascal's Theorem.

In the more general case, if P lies inside one of s, t or if it lies outside both, but one of the tangents from P to either of these conics meets the other in a pair of distinct real points, there must be lines through P which meet both s and t in real points. Four more points of the required conic can be constructed by finding the mate of P in the involution determined on a line through P by the pairs of points in which that line meets s and t. The required conic may then be constructed by Pascal's Theorem.

If neither tangent from P to each of s, t meets the other in real points, then s, t must lie entirely outside each other, and so have four distinct real common tangents.

In this case a real common self-polar triangle of s, t can be constructed; and three other points Q, R, S on the required conic are known, on the lines joining P to the vertices of this triangle, being the harmonic conjugates of P with respect to a vertex and a point on the opposite side. The tangent at P is also determined, for it passes through the point conjugate to P for the pencil, and this, by Art. 210, is the intersection of the polars of P for s and t. We have now four points P, Q, R, S and the tangent at one of them P, and the conic can be constructed as in Chapter V.

The last construction holds even if P lies on a common tangent to s, t. If, however, P lies on *two* common tangents to s, t, then P lies on a side EF of the common self-polar triangle EFG, and only one other point Q is obtained from the above construction; this point Q is the harmonic conjugate of P with respect to E, F and is the meet of the other two common tangents of s, t. Since the points of contact of s, t with any one of their common tangents are double points of the involution determined by the pencil of conics on these tangents (Art. 207) the four harmonic conjugates of P, Q with respect to these points of contact are points on the required conic, on which we now have six points.

In the special case where the conics s, t touch externally at A, and P lies on their common tangent at A, the required conic must touch AP at A and pass through P. It must therefore break up into a line-pair, one component of which is AP. If now any straight line u be drawn, which meets both s and t at real points, and also meets AP at Q, then the mate of Q in the involution in which the intersections of u with s and t are pairs of mates is a second point Q' of the line pair, lying on the component other than AP. By taking a second position of u we find a second point Q' and the join of these points Q' gives the second component of the line-pair.

If s, t have double contact externally and P lies on both common tangents, the conic through P clearly reduces to the line-pair formed by these common tangents.

EXAMPLE

Show how to construct the conic of a range, which touches a given straight line, when two conics of the range are given.

216. Conics having double contact. When two conics touch at A and B they define a pencil of conics touching the two given conics at A and B. The pole E of AB is the same for all the conics; and if F, G be any pair of points on the common chord of contact harmonically conjugate with regard to A, B, EFG is a common self-polar triangle of the pencil of conics. There is thus an infinity of common self-polar triangles. The three line-pairs of the system degenerate into the doubled line AB, occurring twice over, and the pair of common tangents EA, EB.

Also such a pencil of conics may be looked upon as forming a range, the four common tangents being coincident in pairs.

Such a set of conics possesses the properties both of the pencil and of the range. Thus to any point there is a conjugate point and to any line a conjugate line, with regard to all the conics of the set.

Hence the locus of poles of any straight line with regard to the conics of the set is a straight line and the polars of a point pass through a point. It is of interest to see how these occur as degenerate cases of the eleven-point and eleven-line conic respectively.

Consider any point Q (Fig. 70) on a straight line q. Let q meet AB at R and let R' be the harmonic conjugate of R with respect to

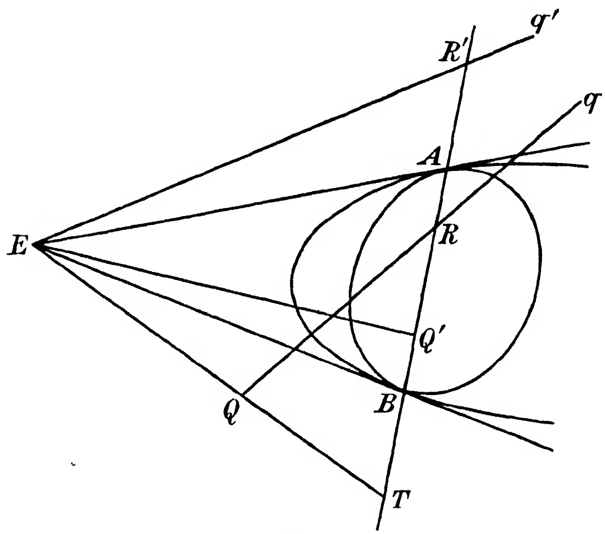


FIG. 70.

A, B. Since R, R' are conjugate with regard to all conics of the set, and E, R are conjugate with regard to all conics of the set, R is the pole of ER' ($=q'$) with regard to all the conics of the set. And since q passes through R, q' is the locus of poles of q with regard to the conics of the set. On the other hand consider the ray harmonically conjugate to EQ with regard to EA, AB. Let it meet AB at Q', and let EQ meet AB at T. Then $Q'ET$ is a self-polar triangle for all the conics of the set, or Q' is conjugate to Q with regard to the set of conics.

The eleven-point conic corresponding to q therefore breaks up into two straight lines, of which one AB is the locus of points conjugate to points of q, and the other ER' is the locus of poles.

Thus these two loci, which are the same in the general case, are now separated. In like manner the eleven-line conic corresponding to Q breaks up into the point E which is the envelope of lines conjugate to lines through Q, and the point Q' which is the envelope of polars of Q with regard to the set of conics.

EXAMPLES

1. If E is the common centre of a set of concentric circles, prove directly: (i) that P is conjugate to the point P'^{∞} at infinity in the direction perpendicular to EP, for every circle of the set; (ii) that any line u is conjugate for every circle of the set, to the line u' perpendicular to u through E.

Hence deduce the results of Art. 216.

2. Prove that the product of pencils of first order conjugate for a pencil of conics having double contact at A, B is a conic passing through A and B.

217. Construction of conics through three points and touching two lines. Let it be required to construct a conic to pass through

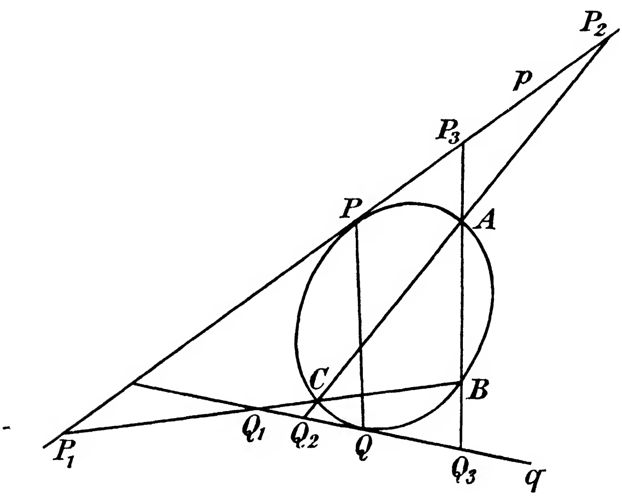


FIG. 71.

three points A, B, C and to touch two lines p, q (Fig. 71). Let the conic required touch p, q at P, Q respectively.

Consider the involution determined on BC by the pencil of conics having contact with p and q at P and Q. If p, q meet BC at P_1, Q_1 then P_1, Q_1 are mates in this involution, for the pair p, q is a conic of the pencil. Also B, C are mates in this involution. The double points of this involution are therefore determined. But since PQ

doubled is a conic of the pencil, the point where PQ meets BC is one of the double points of this involution.

In like manner PQ passes through one of the double points of the involution on AC determined by the pairs of mates (A, C) (P_2, Q_2); P_2, Q_2 being the points where p, q meet AC.

There are thus four possible positions of PQ corresponding to the four lines joining the double points of these two involutions, and so there are four solutions to the problem proposed.

The reader may verify that if PQ passes through double points of the involutions on BC, CA, it will also pass through a double point of the corresponding involution on AB.

Reciprocating the above construction we obtain a construction for the conics through two points and touching three lines. This, like the above, has in general four solutions.

Examples

1. Prove that the problems: to draw a conic touching two given real lines and passing through three given real points, and to draw a conic touching three given real lines and passing through two given real points have either four real solutions or none.

2. By projecting the circular points at infinity into any two conjugate imaginary points, prove that there are always four real conics passing through two conjugate imaginary points and touching three real lines.

218. Properties of confocal conics. If two of the opposite vertices of the quadrilateral *abcd* are the circular points at infinity Ω, Ω', the range of conics inscribed in this quadrilateral becomes a system of confocal conics. The involution of tangents through any point P has thus the circular lines through P for mates. Its double rays are therefore at right angles (Art. 141), and they bisect the angles between any pair of mates (Art. 101). Such a pair of mates are the lines joining P to the two real foci S, S'. We get the series of theorems:

Through any point P of the plane two conics of a confocal system can be drawn and these cut at right angles.

The tangent and normal at any point P of a conic bisect the angles between the focal distances.

The two tangents from P to a conic are equally inclined to the rays joining P to the real foci.

Also, from the property above that double rays are at right angles, conjugate lines with regard to a system of confocal conics are perpendicular. Hence:

The locus of the poles of any straight line q with regard to a system of confocals is the normal at Q to the conic of the system touching q, Q being the point of contact of q with this conic.

The theorem of Art. 181 that coaxal circles reciprocate, with respect to a limiting point into confocal conics, has an instructive interpretation from the point of view of pencils and ranges of conics.

Coaxal circles are clearly a special case of a pencil of conics, since they pass through Ω, Ω' and through two other fixed points, say A and B.

The three line-pairs of the system are

$$(AB, \Omega\Omega'), (A\Omega, B\Omega'), (A\Omega', B\Omega).$$

The first consists of the radical axis and the line at infinity; the last two are the circular lines through C and D, where C and D are the points $(A\Omega, B\Omega')$ and $(A\Omega', B\Omega)$ respectively, that is, they are by Art. 140 point-circles at C and D.

The points C, D are the limiting points of the system of coaxal circles. They are imaginary if A, B are real, but real if A, B are conjugate imaginary, that is, if the radical axis does not cut the circles in real points (*cf.* Art. 113).

Consider now the effect of taking polar reciprocals of the coaxal circles with regard to any circle of centre C. We obtain a range of conics touching the four polars of A, B, Ω, Ω' with regard to such a circle.

Now $C\Omega$ being the tangent at Ω to the circle whose centre is C, the pole of $C\Omega$ with regard to this circle is Ω. Hence the polar of A (which lies on $C\Omega$) passes through Ω. Similarly the polar of Ω' is $C\Omega'$ and the polar of B passes through Ω'. Thus A, B, Ω, Ω' reciprocate into lines ΩF, $\Omega' F$, ΩC, $\Omega' C$. The circles therefore reciprocate into conics having C, F for foci.

219. Properties of rectangular hyperbola. If two conics of a pencil are rectangular hyperbolas the points Ω, Ω' are conjugate with regard to two conics of the pencil. Therefore they are conjugate with regard to all the conics of the pencil. These are therefore all rectangular hyperbolas. Thus every conic through the intersections of two rectangular hyperbolas is a rectangular hyperbola.

If A, B, C, D be the four intersections of two rectangular hyperbolas, the line-pairs are also rectangular hyperbolas, therefore they are perpendicular. The quadrangle $ABCD$ is therefore such that pairs of opposite sides are perpendicular. Any one of its four vertices is the orthocentre of the triangle formed by the other three. It follows that any conic through the three vertices of a triangle and its orthocentre is a rectangular hyperbola (*cf.* Art. 93). Conversely the orthocentre of any triangle inscribed in a rectangular hyperbola lies on the curve. For if ABC be the triangle and the perpendicular through A to BC meet the hyperbola again at D, the pair AD, BC being a rectangular hyperbola, every conic through A, B, C, D is a rectangular hyperbola. But CA, BD is such a conic, therefore CA, BD are perpendicular, or D is the orthocentre of ABC.

220. Centre loci. The theorems of Arts. 211, 212 give the following results when q is the line at infinity.

The locus of the centres of a pencil of conics through four points A, B, C, D is a conic whose asymptotes are parallel to the axes of the two parabolas through the four points and which passes through the vertices of the diagonal triangle of the quadrangle $ABCD$ and the middle points of the six sides of this quadrangle.

Incidentally we have proved the theorem :

The six middle points of the sides of a complete quadrangle lie on a conic which circumscribes its diagonal triangle.

The locus of the centres of a range of conics touching four lines a, b, c, d is a straight line. Since the mid-points of the three point-pairs are evidently centres, they lie on this locus. Hence, incidentally : the middle points of the three diagonals of a quadrilateral are collinear.

EXAMPLES

1. Show how to construct the centre of a conic touching five given lines.

2. The centre of the locus of centres of conics of a pencil is the centroid of the quadrangle defining the pencil.

[For the centre-locus passes through the mid-points P, Q, R, S of AB, BC, CD, DA. But $PQRS$ is readily shown to be a parallelogram. Hence the intersection of PR, QS, which is the centroid of the quadrangle, is also the centre of the centre-locus.]

3. Prove that the asymptotes of any conic of a pencil are parallel to harmonic conjugates with respect to the asymptotes of the centre-locus of the pencil.

4. The conics of a pencil touch a given straight line l at a point A of it, and pass through two other points B and C. Show that the centre-locus passes through A, the middle points D, E, F of BC, CA, AB respectively and the meet H of BC and l ; and also that the tangent at A is harmonically conjugate to l with respect to AC, AB, and the tangents at E and F are parallel to l.

5. The conics of a pencil have four-point contact at a point A. Prove that their centres lie on a line through A.

221. Locus of foci of conics of a range. The involutions of tangents from two different points P, Q of the plane to the conics of a range are clearly homographically related, since either tangent through P determines uniquely the conic of the range and therefore the pair of tangents from Q: and the converse is true if we start from Q.

These homographic involution pencils, however, have a self-corresponding ray, namely PQ, for PQ is a tangent from either P or Q to the conic of the range which touches PQ.

The locus of intersections of tangents from P and Q therefore reduces, by Art. 203, to a cubic through P and Q.

If now P and Q be taken at Ω, Ω' this locus becomes the locus of the foci of the conics of the range.

The foci of the range therefore lie on a cubic through Ω, Ω'. Such a cubic is known as a circular cubic.

If P and Q lie on any one of the three diagonals of the quadrilateral *abcd* which defines the range, the tangents from P to the corresponding point-pair coincide along PQ; and so do the tangents from Q to the same point-pair. The homographic involutions from P and Q have therefore a self-corresponding double ray. Their product therefore reduces to a conic with regard to which P and Q are conjugate (Exs. XIIA, 13).

If P, Q be Ω, Ω', the corresponding diagonal of the quadrilateral is at infinity; the quadrilateral is a parallelogram. Hence the locus of the foci of all conics inscribed in a parallelogram is a rectangular hyperbola (for Ω, Ω' are conjugate with regard to it, by the above).

If the quadrilateral, instead of being a parallelogram, is symmetrical about a diagonal, this diagonal is obviously part of the locus. Since it does not pass through Ω, Ω' the cubic breaks up into this diagonal and a conic through Ω, Ω', that is, a circle.

Since (*see* Ex. 2, Art. 119) the components of a point-pair are also its foci the rectangular hyperbola which is the locus of foci of conics inscribed in a parallelogram circumscribes the parallelogram, and the circle which is the corresponding locus for the quadrilateral symmetrical about a diagonal passes through the four vertices not on this diagonal.

222. The hyperbola of Apollonius. Let s be a conic, c any circle in its plane, with centre O; c meets s at four points A, B, C, D. Let k be the centre-locus of the pencil of conics through A, B, C, D. Clearly c meets the line at infinity at Ω, Ω'. Let s and k meet the line at infinity at I^∞, J^∞ and at X^∞, Y^∞ respectively. Then X^∞, Y^∞ are the double points of the involution defined by the pairs of mates I^∞, J^∞ and Ω, Ω'. Thus X^∞, Y^∞ correspond to perpendicular directions harmonically conjugate to those of I^∞, J^∞, that is, they bisect the angles between the latter, namely between the asymptotes of s : hence they are parallel to the axes of s. Accordingly k is a rectangular hyperbola passing through the centre of s and the point O ; it is called the **hyperbola of Apollonius** for O.

Now k meets s at four points P, Q, R, S. At one of these four points, say P, draw the tangent to s, meeting the line at infinity at T^∞. By the property of the centre-locus, the point conjugate to P for the pencil of conics through A, B, C, D is on the line at infinity, and since PT^∞ touches a conic of the pencil, the point conjugate to P for the pencil also lies on PT^∞. Hence it must be T^∞.

Since T^∞ is conjugate to P for all conics through A, B, C, D, it is conjugate to P for the circle c. Hence the polar of T^∞ with regard to the circle, that is the line through O perpendicular to the direction of T^∞, passes through P, and OP is normal to s at P. Similarly OQ, OR, OS are normals to s.

Thus **from any point O four normals, real or imaginary, can be drawn to the conic, of which the feet are the intersections of the conic with the hyperbola of Apollonius for O.**

Note that any hyperbola k whose asymptotes are parallel to the axes of s and which passes through the centre C of s is a hyperbola of Apollonius for some point of s. For let P be any one intersection of k and s ; let the normal at P to k meet k again at O. Then the hyperbola of Apollonius for O passes through O, C and P, and also the two points at infinity on the axes. It is therefore identical with the hyperbola k. Accordingly any such hyperbola meets the conic s at four points the normals at which are concurrent at a point of the hyperbola.

Examples

1. Prove that, if a circle centre O meet a conic s at four points A, B, C, D, the vertices of the diagonal triangle of the quadrangle $ABCD$ lie on the hyperbola of Apollonius for O and s.

2. Show that the pencils of conics defined by a conic s and any of a set of circles with a common centre O have the hyperbola of Apollonius for O and s as their common centre-locus.

3. By taking the circle with centre O as the line-pair $O\Omega$, $O\Omega'$, prove that the Frégier point of O for the hyperbola of Apollonius for O and a conic s is the point at infinity on the polar of O with respect to s.

4. Show that the tangent at O to the hyperbola of Apollonius for O and s is perpendicular to the polar of O with respect to s.

5. Show that, if s be a conic whose centre is C, and O is any point, the

tangent at C to the hyperbola of Apollonius for O and s is the diameter of s conjugate to the direction perpendicular to OC.

6. The feet of the perpendiculars from a point O on the axes of an ellipse are M, N, and the perpendicular from O on the diameter conjugate to CO meets MN at L. Show that the point K harmonically conjugate to L with respect to M, N is the centre of the hyperbola of Apollonius for O.

223. Joachimsthal's Theorem. Let L, M, N, K (Fig. 72) be the feet of four concurrent normals to a conic s. Consider the involution determined on the axis AA' by the pencil of conics through $LMNK$. The line-pair LM, NK determines two points P, P'. The conic s determines A, A'. The hyperbola of Apollonius, having its asymptotes parallel to the axes of s and passing through C

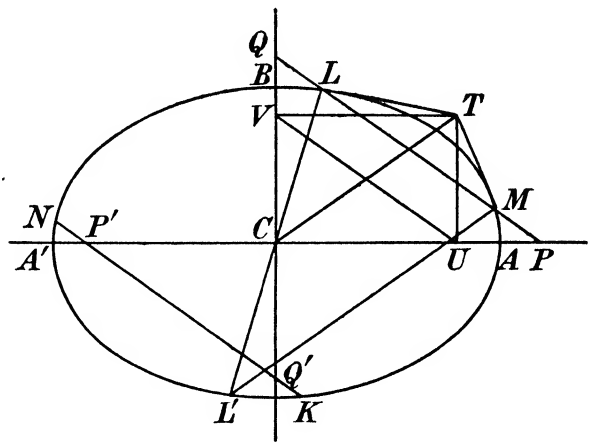


FIG. 72.

determines C and the point at infinity on AA'. C is then the centre of this involution. Thus

$$CP.CP' = -CA^2 \quad \text{.....................(1)}$$

But if T be the pole of LM with regard to s and TU be drawn perpendicular to AA', TU is the polar of P. U, P are therefore mates in the involution on AA' of conjugate points with regard to s: and A, A' are double points in this involution.

Hence
$$CP.CU = CA^2 \quad \text{........................(2)}$$

From (1) and (2)

$$CP' = -CU.$$

Similarly if TV be drawn perpendicular to the other axis CB and NK meet this axis at Q'

$$CQ' = -CV.$$

Hence $P'Q'$, *i.e.* NK, is parallel to VU. But VU and CT, being diagonals of the rectangle $CUTV$, are equally inclined to the axes CA, CB. Therefore NK and CT are equally inclined to the axes. But CT is the diameter of s conjugate to the direction of LM. If L' be the other extremity of the diameter CL, $L'M$ (being a supplemental chord to LM) is parallel to CT and therefore equally inclined with NK to the axes. Hence by Art. 77 a circle will go through L', M, N, K. This is Joachimsthal's Theorem, that **if four normals to a conic be concurrent, the circle through the feet of three of them passes also through the point diametrically opposite to the foot of the fourth.**

224. Geometrical constructions for transforming any two conics into conics of given type. We have already seen (Art. 149) how to transform any two conics into circles.

Any two conics may be transformed into concentric conics by a real projection. For we have seen that there is always one side of the common self-polar triangle which is real. Projecting that side to infinity the opposite vertex projects into the common centre.

If two vertices of the common self-polar triangle of two conics be projected into Ω, Ω', the conics project into concentric rectangular hyperbolas.

Any two conics which do not touch may be projected into coaxial conics. Thus: if EFG be their common self-polar triangle, project FG to infinity and the angle FEG into a right angle.

Two coaxial conics s_1, s_2 can be transformed into one another by reciprocal polars.

Let C (Fig. 73) be their common centre, A_1A_1' and B_1B_1' the axes of s_1, A_2A_2' and B_2B_2' the axes of s_2. Find the double points A, A' of the involution determined by the pairs of mates (A_1, A_2) (A_1', A_2') and the double points B, B' of the involution determined by the pairs of mates (B_1, B_2) (B_1', B_2'). Then from symmetry about C a conic s exists having AA', BB' as axes. Form the reciprocal polar conic s_2' of s_1 with regard to s. s_2' passes through A_2, A_2', B_2, B_2', and its tangent at A_2, being the polar of A_1 with regard to s, is perpendicular to CA_2 and so is the same as the tangent to s_2 at A_2. s_2', s_2 are thus identical.

Clearly either extremity of each axis of s_1 may be denoted by an accented letter: hence the above construction can be carried out in four separate ways. Projecting back, we see that there are four conics with respect to which the original conics are reciprocal polars.

We have thus proved that, if k_1, k_2 are two conics with four distinct intersections P, Q, R, S, and therefore also four distinct common tangents p, q, r, s, there exists at least one conic k with respect to which k_1, k_2 are reciprocal polars. Thus k is a conic with respect to which P, Q, R, S are the poles of p, q, r, s; this really gives more than enough conditions to determine k, but these are necessarily consistent. It will be found that there are four ways of correlating P, Q, R, S with p, q, r, s; these are settled by the consideration that pq lies on the side of the common diagonal triangle opposite to the vertex through which PQ passes.

If now one or more of the points P, Q, R, S (and therefore also of

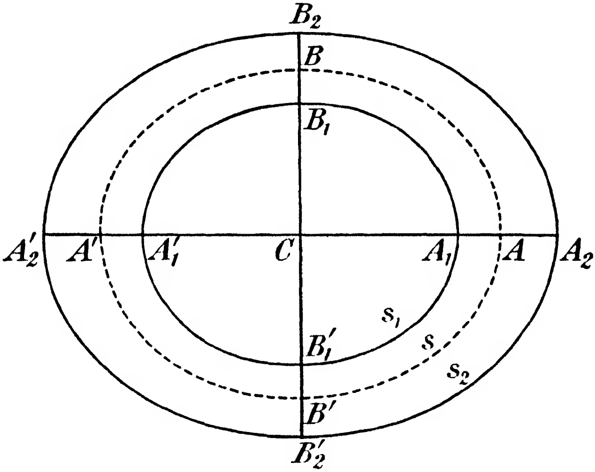


FIG. 73.

the tangents p, q, r, s) are made to approach one another, the above proposition will still hold good in the limiting cases, provided enough conditions are left to determine the conic k, and in this way we can take into account the case of conics in contact, which cannot be transformed into coaxial central conics.

Thus, if two conics touch at O, x being the common tangent, and if P, Q be their other common points, p, q their other common tangents, the conic k is now such that it has (P, p) and also (Q, q) as pole and polar, and touches x at O. This is in fact more than enough to determine it, but these conditions can be shown to remain consistent, by proceeding to the limit from the more general case.

If the conics have three-point contact at O, they have only one other common point P and one other common tangent p. The conic k is now determined from the condition that it has three-point contact with the given conics at O, and that P is the polar of p with regard to it. These are just enough conditions to fix k.

If the conics k_1, k_2 have four-point contact at O, all we get from the general condition above is that k has also four-point contact at O with k_1, k_2, and this is not enough to determine k. In this case, however, we can proceed as follows. Let a given ray through O meet k_1, k_2 at A_1, A_2 and let A be harmonically conjugate to O with respect to (A_1, A_2). Let the conic k be taken through A. Then, since conics having four-point contact at O are in plane perspective, O being the pole and the common tangent x at O the axis of perspective (Art. 46), if an arbitrary ray through O meet k_1, k_2, k at P_1, P_2, P we have that A_1P_1, A_2P_2, AP meet on x, and $\{OP_1PP_2\}=\{OA_1AA_2\}=-1$. Hence the polar of P_1 with respect to k passes through P_2. But the tangents at P_1, P_2, P concur at a point T of x, which is the pole of OP with respect to k. Since P_1 lies on OP, the polar of P_1 with respect to k passes through T; therefore it is TP_2 and touches k_2. Hence k_1, k_2 are polar reciprocals with respect to k.

Finally there is the case of conics having double contact. These can be projected into concentric circles of radii a_1, a_2. Applying now the construction of Fig. 73, where *any* pair of perpendicular diameters can now be taken as axes, the circles are polar reciprocals with respect to: two concentric circles of radii $\sqrt{\pm a_1a_2}$ and two concentric conjugate rectangular hyperbolas, whose semi-axes are $\sqrt{\pm a_1a_2}$. If we vary the axes, the circles remain the same, but there is an infinity of rectangular hyperbolas, which are possible conics k.

In every case a conic k exists for which two given conics are polar reciprocals, though such a conic is not necessarily real.

Example

Prove that two conics which have simple contact may be projected into two coaxial parabolas, but that if they have three-point or four-point contact this is not possible.

225. Two triangles self-polar for the same conic. Let ABC, $A'B'C'$ be two triangles self-polar for the same conic k. Then we have AB, AC, AB', AC' are conjugate to $A'C$, $A'B$, $A'C'$,

$A'B'$ respectively. Hence, since conjugate pencils through A, A' are projective

$$A(BCB'C') \barwedge A'(CBC'B') \barwedge A'(BCB'C'),$$

by double interchange.

But the above is the condition that B, C, B', C' lie on a conic s through A and A'.

Hence, **if two triangles are self-polar for a conic k, their six vertices lie on a conic s.**

Reciprocate the above theorem with regard to the conic k, the self-polar triangles reciprocate into themselves, the vertices

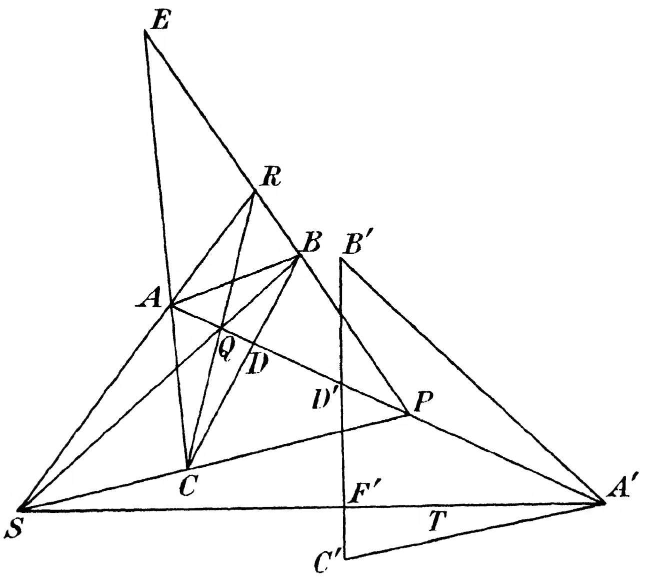


FIG. 74.

reciprocating into the sides, so that the six sides of the triangle touch the conic s', which is the reciprocal of s with respect to k.

Thus, **if two triangles are self-polar for a conic k, their six sides touch a conic s'.**

226. Two triangles inscribed in a conic are self-polar for a conic. We now proceed to prove the converse of the theorem of the last Article.

Let ABC, $A'B'C'$ (Fig. 74) be two triangles inscribed in a conic s.

We shall first show that a unique conic k exists for which ABC is self-polar and A' is the pole of $B'C'$. Let us first assume that such a conic exists. Join AA', and let it meet BC, $B'C'$ at D, D' respectively. Then (A, D) and (A', D') are pairs of conjugate points for k on the line AA'. Let P, Q be the double points of the involution on AA' of which (A, D) and (A', D') are pairs of mates. Then P, Q are the points at which AA' meets k.

Let $(BP, CQ)=R$, $(BQ, CP)=S$, and $(BP, CA)=E$. Since A, D are harmonically conjugate with respect to P, Q, then, by the property of the complete quadrangle $BCPQ$, the side RS of the diagonal triangle passes through A. Also, since $\{PAQD\}=-1$, $C\{PAQD\}=-1$, and, cutting by the transversal BP, $\{PERB\}=-1$. Therefore, since B, E are conjugate for k (ABC being self-polar for k), and P lies on k, R lies on k. Similarly S lies on k.

Hence k belongs to the pencil of conics through P, Q, R, S. Join $A'S$ meeting $B'C'$ at F', and let T be the harmonic conjugate of S with respect to A', F'. Since $B'C'$ is the polar of A' with respect to k, A', F' are conjugate for k, and T must be a point of k.

Conversely, the conic through P, Q, R, S, T, which is uniquely determined, satisfies all the conditions for k.

For (i) ABC, being the diagonal triangle of a quadrangle $PQRS$ inscribed in this conic, is self-polar for it; (ii) A' is conjugate for this conic to both D' and F', and therefore is the pole of $D'F'$, that is, of $B'C'$. Hence $PQRST$ is the conic k required; this conic accordingly exists and is unique, though not necessarily real.

Let now, if possible, the point of $B'C'$ conjugate to B' for k be some point C'' other than C'. Then $A'B'C''$ is self-polar for k. Hence, by Art. 225, A, B, C, A', B', C'' lie on a conic. But this last conic passes through five points of s, namely A, B, C, A', B', and so is identical with s. Hence s passes through both C' and C'', so that, if these were distinct, $B'C'$ would meet s in three points, which is impossible. Thus C'' and C' coincide and the triangles ABC, $A'B'C'$ are self-polar for k.

Reciprocating this theorem we see that **if two triangles are circumscribed to a conic, they are self-polar for a conic.**

In the above, and in Art. 225, it has been assumed that the triangles in question do not have either a common vertex or a common side. The theorems are, however, capable of interpretation even in this case. Thus, if $A=A'$, the triangles ABC, $AB'C'$ are self-polar for the line pair x, y where x, y are the double rays of the involution pencil of which (AB, AC) and (AB', AC') are pairs of mates. Similarly if BC, $B'C'$ are in a line, the triangles are self-polar for the point-pair X, Y where X, Y are the double points of the involution determined by (B, C) and (B', C'). In this last case the conic on which A, B, C, A', B', C' lie itself degenerates into the line-pair AA', BC.

EXAMPLE

Prove that only one circle can be drawn for which a given triangle is self-polar.

227. Outpolar and inpolar conics. If a triangle ABC self-polar for a conic k is inscribed in a conic s, then there exist an infinite number of such triangles, one vertex of which may be selected arbitrarily upon s.

Take any point A' of s and let the polar of A' with respect to k meet s at B', C'. Then ABC and $A'B'C'$ are self-polar for a conic k, for which ABC is self-polar and A' is the pole of $B'C'$. But this conic k, by Art. 226, is uniquely determined, and is therefore identical with the conic k. Hence $A'B'C'$ is inscribed in s and self-polar for k.

The conic s, which is such that a triangle inscribed in s is self-polar for k, is said to be **outpolar** to k.

Similarly, if a triangle abc self-polar for k is circumscribed to s' there exist any number of such triangles, and any tangent to s' may be taken as a side of such a triangle, which is then determined. The conic s' is then said to be **inpolar** to k.

We have seen in Art. 224 that it is always possible to transform one conic into another by reciprocal polars. If t be the conic with respect to which s and k are reciprocal, then the triangle ABC which is self-polar for k, reciprocates with respect to t into a triangle $A_1B_1C_1$ which is self-polar for s. Also the points A, B, C of s reciprocate into lines a_1, b_1, c_1 which touch k, and form the sides of the triangle $A_1B_1C_1$. Thus $A_1B_1C_1$ is self-polar for s and circumscribed to k. Hence k is inpolar to s.

Similarly, if s' and k are reciprocated into one another, it is found that k is outpolar to s'.

The relations of outpolarity and inpolarity are therefore reciprocal, so that if a conic is outpolar to a second conic, then the second is inpolar to the first, and conversely.

EXAMPLES

1. Prove that, if a conic s_1 is outpolar to s_2 and P, Q are two points of s_2, which are conjugate for s_1, then the pole R of PQ for s_2 lies on s_1, and conversely, that the polar with regard to s_2 of a point of s_1 meets s_2 at points conjugate for s_1.

2. Prove that, if s_1 is inpolar to s_2, and p, q are two tangents to s_2, which are conjugate for s_1, then their chord of contact r touches s_1, and, conversely, if R be the pole with regard to s_2 of a tangent r to s_1, the tangents from R to s_2 are conjugate for s_1.

3. Prove that the locus of the centre of a rectangular hyperbola which is inscribed in a given triangle is the circle for which the triangle is self-polar.

[For the hyperbola is inpolar to the circle, *i.e.* triangles self-polar for the hyperbola are inscribed in the circle, $\therefore$ taking Ω or Ω' as a vertex of such a triangle, the centre of the hyperbola lies on this circle.]

4. Prove that all circles through the centre of a rectangular hyperbola are outpolar to the hyperbola.

A circle is drawn to pass through the centre of a rectangular hyperbola, and P is the pole of an asymptote with respect to the circle. Show that the tangents from P to the hyperbola are harmonically conjugate with respect to the tangents from P to the circle.

5. If a line-pair have its double point on a conic, show that the conic is outpolar to the line-pair; also that, if the line joining the points of a point-pair touches a conic, the conic is inpolar to the point-pair.

6. Prove that, if a line-pair is outpolar to a conic, the lines of the pair are conjugate for the conic, and that, if a point-pair is inpolar to a conic, the points of the pair are conjugate for the conic.

228. Conic triangularly inscribed in a conic. Poncelet's Porism. Since two triangles ABC, $A'B'C'$ inscribed in a conic s are self-polar with respect to a conic k (Art. 226), and two triangles self-polar with respect to a conic k are circumscribed to a conic s' (Art. 225), it follows that, **if two triangles are inscribed in a conic, they are circumscribed to another conic.** For a direct proof of this theorem, *see* Exs. IIIA, 17.

It follows that if one triangle ABC can be inscribed in s and circumscribed to s', any number of such triangles exist.

For take any point A' on s and from A' draw the two tangents $A'B'$, $A'C'$ to s', meeting s again at B', C' respectively. Then, since ABC, $A'B'C'$ are inscribed in s, their sides touch a conic t. But s' and t have five tangents the same, namely BC, CA, AB, $A'B'$, $A'C'$. Hence t coincides with s' and $B'C'$ touches s'.

We shall then say that s is **triangularly circumscribed** to s', and that s' is **triangularly inscribed** in s.

A similar result is easily shown to hold for tetragons, this being a four-sided figure in the Euclidean sense, whose sides and vertices are taken in order.

For let a tetragon $ABCD$ be inscribed in s, and let its sides AB, BC, CD, DA touch s'. Let $(AB, CD) = X$, $(BC, DA) = Y$, and $(AC, BD) = O$, so that OXY is self-polar for s. Let s meet XY at U, V. Project U, V into the circular points, then, since XY are harmonic for U, V, they project into points at infinity in perpendicular directions, while s projects into a circle s_1. Also O is the pole of XY with respect to s', as well as with respect to s. Hence s' projects into a conic s_1' concentric with the circle, and $ABCD$ projects into a rectangle $A_1B_1C_1D_1$ circumscribed to s_1' and inscribed in s_1. Thus s_1 is the orthoptic circle of s_1'. But if we now draw any two parallel tangents to s_1', and also the two perpendicular tangents, we obtain another rectangle $A_1'B_1'C_1'D_1'$ circumscribed to s_1', whose vertices must lie on the orthoptic circle s_1. Any number of such rectangles can be drawn. Projecting back, we obtain an infinite number of tetragons inscribed in s and circumscribed to s'. We may say, in such a case, that s is **tetragonally circumscribed** to s'.

Nor need this conception be limited to four-sided figures. For it is clear that, if two circles are inscribed and circumscribed to a regular polygon of n sides, any number of such polygons can be obtained by rotating the original one through an arbitrary angle about the common centre. Projecting the circular points into two arbitrary points S, T, we obtain two conics s, s' having double contact at S, T, which possess the property that an infinite number of polygons of n sides can be inscribed in s and circumscribed to s'. We may then speak of s as **polygonally circumscribed,** or n-**gonally circumscribed,** to s'.

The theorems of this Article are particular cases of a more general theorem, due to Poncelet, and known as **Poncelet's Porism,** in which the restriction, which has been introduced above in the case $n>4$, that the conics have double contact, is removed.

Examples

1. Show that a unique conic can be drawn triangularly circumscribed to a given conic and touching it at two given points H, K.

If conics s, s' touch at H, K, and ABC is a triangle inscribed in s and circumscribed to s', show that BC meets HK at the point whose harmonic conjugate with regard to (B, C) is the point of contact D of BC with s'; and show also that DH, DK are harmonically conjugate with regard to DA, DB.

[Project H, K into the circular points.]

2. From the theorem that if two triangles are circumscribed to a conic they are inscribed in another conic prove, by taking two vertices of one triangle to be the circular points at infinity, that the circle circumscribing a triangle formed by three tangents to a parabola passes through the focus.

3. Prove that any circle belonging to either of the coaxal systems which have the real foci of a conic (i) as real intersections, or (ii) as real limiting points, is tetragonally circumscribed to the conic.

229. Polar quadrangles and quadrilaterals. Let $ABCD$ (Fig. 75) be a quadrangle, of which two pairs of opposite sides, $AB=p$ and $CD=p'$, $AD=q$ and $BC=q'$ are conjugate with regard to a conic k.

We will now prove that in this case the third pair of opposite sides, $AC=r$ and $BD=r'$, are also conjugate with respect to k.

Let P, P', Q, Q' be the poles of p, p', q, q'. Then P lies on CD, P' lies on AB, Q lies on BC and Q' lies on AD. Also, since $A=pq$, the polar of A is PQ.

Consider now the ranges of conjugate points on AB, AD (*i.e.* p, q). These are projective by Art. 52, and, since A corresponds

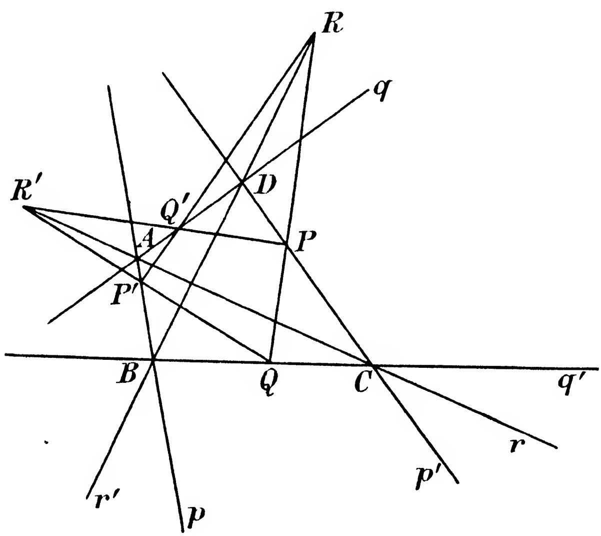


FIG. 75.

to the intersections of p, q with the polar of A, this polar is the cross-axis of the above ranges.

But further, $B=pq'$ has PQ' for its polar, and $D=p'q$ has $P'Q$ for its polar. Hence in the conjugate ranges on p, q P' corresponds to D and B to Q'. Therefore BD, $P'Q'$ meet on the cross-axis, or $P'Q'$, PQ, BD are concurrent. But $(P'Q', PQ)$ is the pole R of $(p'q', pq)$ that is, of AC. Hence the pole of AC lies on BD, and the lines r, r' are conjugate for k. Similarly the pole R' of BD is a point of AC through which pass the lines PQ', $P'Q$.

A quadrangle such as $ABCD$, which is such that any pair of opposite sides are conjugate for a conic k, is said to be a **polar quadrangle** for the conic.

Similarly (or by reciprocation), if two pairs of opposite vertices of a complete quadrilateral are conjugate points, so is the third pair and the quadrilateral is said to be a **polar quadrilateral** for the conic.

A polar quadrangle can clearly be constructed with three of its vertices A, B, C arbitrarily chosen. For, draw through A a line conjugate to BC, and through C a line conjugate to BA. If these lines meet at D, D is the fourth vertex of a polar quadrangle.

Similarly any three sides of a polar quadrilateral can be arbitrarily selected, and the fourth side is then determinate.

Any self-polar triangle ABC forms with an arbitrary fourth point D of the plane a polar quadrangle. For since A is the pole of BC, AD, which passes through A, is conjugate to BC; similarly CD is conjugate to BA and BD is conjugate to AC.

In like manner a self-polar triangle abc forms with an arbitrary line d of the plane a polar quadrilateral.

It should be carefully noted that, in general, two vertices of a polar quadrangle are not conjugate points for the conic.

If, however, two of them, say A and B, are conjugate points then the polar of A, namely PQ (Fig. 75) passes through B. Hence either Q or R coincides with B. Clearly Q, R cannot coincide with one another since q, r are different lines. If R coincides with B, QR coincides with BC, and P with C. Thus A is the pole of BC and C the pole of AB and the triangle ABC is self-polar. If Q coincides with B, QR coincides with BD and P with D: A is the pole of BD, D is the pole of AB, and the triangle ABD is self-polar.

Hence, in such a case, three of the vertices necessarily form a self-polar triangle.

Similarly, if two sides of a polar quadrilateral are conjugate lines, they form with one or other of the two remaining sides a self-polar triangle.

It should be noted that, if three of the vertices A, B, C of a polar quadrangle form a self-polar triangle, the fourth vertex D cannot be conjugate to any one of the three other vertices. For if it were conjugate, say to A, it must lie on BC, and $ABCD$ would no longer be a proper quadrangle.

A similar conclusion holds for the polar quadrilateral formed by three sides of a self-polar triangle and any fourth line, not passing through a vertex of the triangle.

We shall refer to a polar quadrangle, three of whose vertices form a self-polar triangle, as a degenerate polar quadrangle, and similarly to a polar quadrilateral, three of whose sides form a self-polar triangle, as a degenerate polar quadrilateral.

Examples

1. If a conic has A, C for a pair of conjugate points, and B, D for another pair, where AC, BD are the diagonals of a rectangle, prove that the axes of the conic are parallel to the sides of the rectangle.

2. Prove directly that, if two pairs of opposite vertices of a quadrilateral are conjugate pairs for a conic k, then the third pair are also conjugate for k.

230. Polar quadrangles inscribed in a conic. Let now $ABCD$, $ABC'D'$ be two polar quadrangles for a conic k, having two vertices A, B common. By the property of the polar quadrangle (AC, BD) (AD, BC) (AC', BD') (AD', BC') are conjugate pairs. Hence (Art. 52)

$$\begin{aligned} A(CDC'D') &\barwedge B(DCD'C') \\ &\barwedge B(CDC'D') \text{ by Art. 21.} \end{aligned}$$

Therefore C, D, C', D' are intersections of corresponding rays of two projective pencils of four rays through A and B, that is, they lie on a conic passing through A and B.

The above theorem fails if A, B are conjugate points for k, for then (*see* Art. 52) the conjugate relation does not define projective pencils through A and B.

In this case we know by Art. 229 that a third vertex of each quadrangle forms with A and B a self-polar triangle for k. If we call this vertex C, then $C' = C$ and D, D' are any arbitrary points of the plane. The two quadrangles have then three vertices common.

In this case it is still true that both quadrangles are inscribed in a conic, for there are only five vertices, and a conic can always be drawn through five points.

Let now s be any conic circumscribing a quadrangle $ABCD$ polar for k. It is clear from Art. 229 that it is always possible to find two vertices of such a quadrangle which are non-conjugate with regard to k, even when the quadrangle is degenerate. Let C and D be two such non-conjugate vertices. Take any given point A' on s, and complete the quadrangle $A'B_1CD$, self-polar for k. Then B_1 must lie upon s. Also both C and D cannot be conjugate to A', for otherwise the quadrangle would be degenerate, with $A'CD$ as a self-polar triangle, so that C and D would be conjugate for k, which by hypothesis is not the case. Let C be non-conjugate to A'.

If we now take a second given point B' on s, and complete the quadrangle $A'B'CD_1$, polar for k, then D_1 is a point of s.

If now A' is not conjugate to B', let C' be any third given point; complete the quadrangle $A'B'C'D'$ polar for k. Then D' is a point of s, so that a quadrangle $A'B'C'D'$, polar for k, has been inscribed in s, with three arbitrary points A', B', C', as vertices.

If, however, A' is conjugate to B', then the quadrilateral $A'B'CD_1$ is degenerate, and one of C, D_1 is the pole of $A'B'$. Since C is not conjugate to A', D_1 must be the pole of $A'B'$, and $A'B'D_1$ is a self-polar triangle for k, inscribed in s.

If now the third given point C' is not the pole D_1 of $A'B'$, then $A'B'C'D_1$ is a degenerate polar quadrangle for k inscribed in s, with the three given points $A'B'C'$ as vertices. In this case the fourth vertex D_1 is determinate, and only one quadrangle satisfies the conditions.

But if C' is itself the pole D_1 of $A'B'$, then $A'B'CC'$ is the quadrangle required. It may however, in this case, be replaced by any other degenerate polar quadrangle $A'B'C'D'$, where D' is any point of s, other than A', B', or C'. The solution therefore involves an arbitrary element.

It follows, taking all cases together, that if a conic s circumscribe any quadrangle polar for k, it circumscribes an infinity of quadrangles polar for k, in any of which three vertices can be arbitrarily selected on s.

Suppose now we take for B', C' the two points where the polar of A' meets s. Then two vertices (A', B' or A', C') of the polar quadrangle are conjugate. The quadrangle is therefore degenerate, and clearly $A'B'C'$ forms a self-polar triangle. The conic s therefore circumscribes a triangle self-polar for k and therefore is outpolar to k.

Conversely, if s is outpolar to k, we can find a triangle ABC self-polar for k and inscribed in s. This triangle forms with any fourth vertex D lying on s a polar quadrangle for k inscribed in s, so that if any three points A', B', C' are arbitrarily taken on s, not being vertices of a self-polar triangle for k, the fourth vertex D' of the polar quadrangle $A'B'C'D'$ lies on s.

Similarly it can be proved: (1) that two polar quadrilaterals for k, having two non-conjugate sides common, touch a conic t, inpolar to k; (2) that any three tangents a, b, c to t being given, not forming a triangle self-polar for k, the fourth side d of the quadrilateral $abcd$, polar for k, touches t.

231. Relations between triangularly inscribed, outpolar and inpolar conics. Let s_1, s_2 be two conics such that s_2 is triangularly inscribed in s_1. Let ABC, $A'B'C'$ be two of the triangles inscribed in s_1 and circumscribed to s_2. By Art. 226 a conic k exists with regard to which ABC and $A'B'C'$ are both self-polar. Hence s_1 is outpolar to k and s_2 is inpolar to k.

If now PQR be any other triangle which has any two of the following properties: (1) it is inscribed in s_1; (2) it is circumscribed to s_2; (3) it is self-polar for k, then it possesses also the third property.

Take first the case where PQR is self-polar for k and inscribed in s_1.

Apply a transformation by reciprocal polars with k as base conic. Since ABC, $A'B'C'$ are self-polar for k, they transform into themselves. Therefore the conic s_1 through their six vertices transforms into the conic touching their six sides, that is, into s_2. Further, PQR being also self-polar for k, transforms into itself, and the vertices P, Q, R lying on s_1 transform into the opposite sides touching the reciprocal conic s_2. Thus PQR is circumscribed to s_2.

A similar argument shows that if PQR is self-polar for k and circumscribed to s_2, it is inscribed in s_1.

Again, let PQR be inscribed in s_1 and circumscribed to s_2. Construct the self-polar triangle for k which is inscribed in s_1 and has P as its vertex (Art. 227). Let it be $PQ'R'$. By the previous results $PQ'R'$ is circumscribed to s_2. Hence PQ', PR' are the tangents from P to s_2, and so are identical with PQ, PR. Thus $Q=Q'$, $R=R'$ and PQR is self-polar for k.

An important result in this connection is that the common self-polar triangle XYZ of s_1, s_2 is also self-polar for k. For, taking reciprocal polars with respect to k, as before, XYZ reciprocates into a triangle self-polar with respect to s_2, s_1, that is, into itself. It is therefore self-polar for k. Hence XYZ and any triangle ABC inscribed in s_1 and circumscribed to s_2 are both self-polar for k, so that the vertices X, Y, Z, A, B, C lie on a conic.

If in the above the conics s_1, s_2 are coaxial, their common self-polar triangle is formed by the common axes and the line at infinity. If then ABC is a triangle inscribed in s_1 and circumscribed to s_2 a conic can be drawn through A, B, C passing through the centre of s_1 and having its asymptotes parallel to the axes. But this is a hyperbola of Apollonius for s_1. Hence if such a triangle exists the three normals to s_1 at its vertices A, B, C are concurrent.

232. Conics outpolar (or inpolar) to the same conic. If two conics s_1, s_2 are outpolar to the same conic k then their four points of intersection form a quadrangle polar for k. For let A, B, C, D be these four points. Consider the quadrangle polar for k having A, B, C for three vertices. In general its fourth vertex D_1 is uniquely determined, and it must lie on both s_1 and s_2, so that it is identical with D. If, however, ABC is itself a triangle self-polar for k, D_1 is arbitrary, but if we take it at D, we still have a quadrangle polar for k. The result therefore holds in all cases.

It follows that, in general, conics through three given points A, B, C, which are outpolar for k, form a pencil of conics through the four points A, B, C, D, where D is the fourth vertex of the quadrangle $ABCD$ polar for k. We have an exception when ABC is a triangle self-polar for k, when every conic circumscribing ABC is outpolar to k.

In a similar manner, the four common tangents to two conics inpolar to k form a quadrilateral polar for k and conics touching three given lines and inpolar to a given conic form, in general, a range of conics inscribed in the quadrilateral polar for k, of which the three given lines are sides.

It follows from the above that through four given points A, B, C, E of the plane, one conic s can, in general, be described, outpolar to a given conic k. For complete the quadrangle $ABCD$ polar for k. Then the conic through A, B, C, D, E satisfies the conditions, and, in general, is the only conic which does so.

In like manner there exists, in general, one conic through three given points A, B, C which is outpolar to two given conics k_1, k_2. For, complete the quadrangles $ABCD_1$, $ABCD_2$ polar for k_1, k_2 respectively. The conic through A, B, C, D_1, D_2 is, in general, the only one satisfying the conditions.

In like manner, one conic inpolar to k can in general be drawn to touch four given lines, and one conic inpolar to k_1, k_2, to touch three given lines.

233. Faure and Gaskin's Theorem. If we take two circles outpolar to a conic s their intersections form a quadrangle self-polar for s. Hence their radical axis is conjugate with regard to s to the opposite side of this quadrangle, namely the line at infinity. This radical axis therefore passes through the centre of s. The tangents from the centre of s to all circles outpolar to s are then equal, that is, a circle concentric with s cuts orthogonally every circle outpolar to s. This circle is therefore the locus of the point circles outpolar to s. But a point circle is simply two circular lines, and a line-pair outpolar to s reduces to a pair of conjugate lines for s. Hence the above locus is the locus of points the circular lines through which are conjugate for s, that is, the orthoptic circle of s by Art. 144.

Hence **every circle outpolar to a conic cuts orthogonally the orthoptic circle of the conic.**

EXAMPLES

1. Prove the converse of Faure and Gaskin's Theorem, namely that any circle which cuts orthogonally the orthoptic circle of a conic is outpolar for the conic.

2. If a circle cut harmonically the sides of a triangle circumscribed to a conic, it cuts orthogonally the orthoptic circle of the conic.

[For the circle is outpolar to the conic.]

3. Show that the orthoptic circles of the conics of a pencil are orthogonal to a fixed circle.

[For the circle circumscribing the common self-polar triangle of the pencil is outpolar to every conic of the pencil.]

234. Nets and webs of conics. The set of conics outpolar to one, two, three or four conics will be said to form a **net** of the fourth, third, second or first grade respectively, and the set of conics inpolar to one, two, three or four conics will be said to form a **web** of the fourth, third, second or first grade respectively.

We have already seen (Art. 232) that a conic outpolar to a given conic can be made in general to pass through four given points in one way only. The condition that a conic is outpolar to a given conic must therefore be equivalent to a linear relation between the coefficients in the equation of the conic, a conclusion which is confirmed by the result already proved in Art. 232 that a conic outpolar to two conics is free to pass through three given points.

Hence a conic belonging to a net of the first grade has already to satisfy four linear relations between its coefficients. One such conic can then be made to pass through any point of the plane. But the intersections of two such conics form a quadrangle polar with regard to each of the four conics determining the net. The conic through any point P and the vertices of this quadrangle is a conic of the net; and it is the only conic of the net through P. The net therefore reduces in this case to a pencil of conics. Similarly the web of the first grade reduces to a range.

A conic belonging to a net of the second grade may in general be made to pass through two given points of the plane, and one belonging to a web of the second grade to touch two given lines of the plane. The corresponding results for the nets and webs of the third and fourth grades have already been obtained.

235. Similar conics. Two coplanar conics are said to be similar and similarly situated if they correspond in a plane perspective of which the axis is at infinity. It follows that the points at infinity of two such conics coincide, or their asymptotes are parallel.

Conversely if two conics s_1, s_2 have their asymptotes parallel they are similar and similarly situated. For let I^∞, J^∞ be their two common points at infinity, t one of their common tangents touching s_1 at P_1, s_2 at P_2. Through P_1, P_2 draw any two parallel lines meeting s_1, s_2 again at Q_1, Q_2 respectively, and let Q_1, Q_2 meet t at O. With O as pole of perspective, the line at infinity as axis of perspective and P_1, P_2 as a pair of corresponding points, construct the conic s_2' in plane perspective with s_1. Then s_2, s_2' have in common the points I^∞, J^∞, Q_2, P_2 and the tangent at P_2. They are therefore identical, that is, s_1, s_2 are similar and similarly situated.

Any two conics may be projected into similar conics by projecting one of their common chords to infinity. Projecting back we see that any two conics correspond in a plane perspective in which any one of their common chords is taken as axis of perspective (*cf.* Exs. IIIA, 11).

EXAMPLES XIIIA

1. Show that the cross-ratio of the flat pencil formed by the polars of a point U with regard to four conics of a pencil is independent of the position of U in the plane.

2. Show that the cross-ratio of the range formed by the poles of a line u with regard to four conics of a range is independent of the position of u in the plane.

3. Prove that the harmonic conjugates of a variable point with regard to four given pairs of points in involution have a constant cross-ratio.

4. Prove that if P, Q are the points of contact of a variable conic of a range with two sides of the base quadrilateral, PQ passes through a fixed point.

5. Prove that the conics for which two given points A, B are the poles of two given lines a, b respectively, form a pencil of conics having double contact.

6. Prove that the product of a pencil of conics and a homographic flat pencil is a cubic curve, and show how to find the tangents to the curve at the vertices of the base quadrangle of the pencil of conics and at the vertex of the flat pencil.

Show that a cubic of this type can in general be made to pass through eight given points of the plane.

7. Show that the locus of points of contact of the tangents drawn from an arbitrary fixed point P to the conics passing through four given points A, B, C, D passes through P, A, B, C, D, through the vertices E, F, G of the diagonal triangle of $ABCD$ and through the intersections of PE, PF, PG with FG, GE, EF respectively. Show further that this locus is a cubic curve.

8. Two confocal conics s and s' are such that there is a triangle ABC inscribed in s and circumscribed about s'; show that the normals to s at A, B, C are concurrent, and that the normals to s' at its points of contact with the sides of ABC are concurrent.

9. O is a fixed point in the plane of a central conic k: prove that the envelope of the polars of O with respect to conics confocal with k is a parabola p, whose directrix is the line joining O to the centre of k.

Prove also that if S is the focus of p, the relation between the points O, S is mutual.

10. Show that in a range of conics two are rectangular hyperbolas and that their orthoptic circles are the point-circles of the system of coaxal orthoptic circles of the range.

11. The locus of centres of rectangular hyperbolas circumscribing a triangle is the nine-points circle of the triangle.

12. Reciprocate the theorems of Art. 219 with respect to (i) one of the vertices of the base quadrangle, (ii) any other point.

13. s_1, s_2 and k are three given conics. Prove that the polar reciprocals of k with respect to the conics of the pencil, of which s_1, s_2 are members, envelop a quartic curve having double points at the vertices of the common self-polar triangle of s_1, s_2.

Show also that three of the above system of polar reciprocals degenerate into line-pairs, every line of which touches the same conic.

[Prove that the envelope is the locus of points conjugate to points of k with regard to the pencil.]

14. Two conics have three-point contact at C; CP and CQ are the diameters of the two conics through C. Prove that PQ passes through the intersection of the tangent at C and the other common tangent.

15. Prove that, if two conics have four-point contact at O and Q is the pole with regard to the second of the tangent at P to the first, O, P, Q are collinear.

16. The conics of a pencil have three-point contact at A with a circle c and pass through a point B. Prove that their centre-locus touches c at A, passes through the middle point C of AB, and touches the line through C parallel to the tangent at A.

Prove also that the circle of curvature to the centre-locus at A touches c externally at A, and that its radius is one-half that of c. Hence show how to construct the centre-locus.

17. The normals at K, L, M, N to an ellipse whose centre is O each pass through the point H. Prove that the locus of the centres of all conics through K, L, M, N is a hyperbola passing through O and having as the tangent at O the line perpendicular to OH.

18. Prove that from any point O in the plane of a parabola three normals can in general be drawn to the curve.

What becomes, in this case, of the hyperbola of Apollonius for O?

19. If a point O describes a normal to a conic the feet M, N, K of the three other normals drawn from O to the conic form a triangle circumscribing a parabola touching the axes.

[If T (Fig. 72) describe LT, the ranges $[U]$, $[V]$, and $\therefore$ $[P']$, $[Q']$ are similar $\therefore$ NK envelops a parabola.]

20. A hyperbola touches a conic k at P, passes through the centre C of k and has its asymptotes parallel to the axes of k. Show that it meets the normal at P at the centre of curvature of k at P.

21. Prove that the eleven-line conic of a point P with respect to a system of confocal conics whose foci are S, H is a parabola touching the common axes, the bisectors of the angle SPH, and the perpendiculars through S, H to SP, HP respectively.

22. Through a fixed point O a conic s is drawn having double contact with a given conic k. Show that the common chord of s, k meets the tangent at O to s on a fixed straight line.

23. $A_1B_1C_1$, $A_2B_2C_2$ are two triangles inscribed in the same conic. Conics s_1, s_2 are described about $A_1B_1C_1$, $A_2B_2C_2$ respectively, having double contact with one another. Show that their common chord of contact touches the conic for which $A_1B_1C_1$, $A_2B_2C_2$ are self-polar.

24. Show that circles passing through a given point, which are outpolar to a given conic, pass through a second fixed point.

25. The tangents to a conic k at P and Q meet at D. Conics p, q pass through D and touch PQ at P, Q respectively. If A, B, C are the remaining intersections of p, q, prove that $ABCD$ is a polar quadrangle for k.

[Show that p, q are outpolar for k.]

26. If triangles exist which are inscribed in a circle c and circumscribed to a circle c' it is necessary and sufficient that the rectangle contained by segments of chords of c through the centre of c' should be numerically equal to twice the product of the radii.

27. Show how, by a real projection, to project two conics which intersect in only two real points into two similar and similarly situated ellipses.

28. From a given point A a variable chord APQ is drawn to a given conic s. Through P, Q and another given point B a conic is drawn similar and similarly situated to s. Prove that this conic passes through a certain fixed point other than B.

[Project the points at infinity on s into the circular points.]

29. If two coaxial conics be such that a triangle exists which is circumscribed to one conic and inscribed in the other, prove that the axes and the sides of the triangle touch a parabola.

30. If P be the pole of a fixed line l with respect to a variable conic of a pencil, and t the tangent at P to the conic of the pencil which passes through P, prove that the envelope of the lines t is a curve of the third class, which touches the six common chords of the conics of the pencil.

31. Prove that in general, the lines which meet three given coplanar conics in three pairs of points in involution form an envelope of the third class, touching each of the eighteen lines which are common chords of two of the conics.

How is this result modified when the three conics have one or more points in common?

32. If ABC be a triangle circumscribed to a conic s, P, Q, R the points of contact with s of the sides a, b, c of ABC, then if QR, RP, PQ meet a, b, c

at L, M, N respectively, L, M, N lie on a straight line u; and AP, BQ, CR meet at a point U.

Show that if, as s is varied, u passes through a fixed point, then U describes a conic and that in this case the conics s touch a fourth fixed line.

33. If s is the harmonic envelope of two conics s_1, s_2, and s' is the reciprocal polar of s with respect to s_1, prove that s' passes through the four intersections of s_1, s_2 and either has no other intersection with s_2, or entirely coincides with it.

What is the relation between s_1, s_2 in the latter case?

State the theorem obtained by reciprocating the above.

EXAMPLES XIIIB

[The axes of co-ordinates are rectangular.]

1. Construct five points on the locus of points conjugate to the points of $y=0$ with regard to the conics through $(0, 0)$, $(0, 2)$, $(4, 1)$, $(5, 3)$.

Find also both asymptotes of this locus in position.

2. ABC is a triangle with $BC=7$ inches, $CA=5$ inches, $AB=4$ inches; D is an internal point of AC such that $AD=2$ inches and E is an internal point of AB such that $AE=3$ inches; F and G are the points of trisection of DE. Obtain enough points or tangents to determine uniquely each of the conics passing through F and G and touching BC, CA and AB.

3. Construct the conic of which the points $(\pm 1, 0)$ are the foci and for which the lines $y=1$, $x+y=3$ are conjugate.

4. The conics of a pencil have three-point contact with the circle

$$x^2-6x+y^2=0$$

at the origin, and pass through the point P $(2, 4)$. Construct (i) the centre of the conic of the pencil which touches the parallel through P to the axis of y, (ii) the intersections of this conic with the line $y=2$.

5. The conics of a pencil have four-point contact with the ellipse

$$\frac{x^2}{1}+\frac{y^2}{4}=1$$

at the point $P\left(\frac{\sqrt{3}}{2}, 1\right)$ on it. Construct (i) the point of contact Q of the conic of the pencil touching the axis of x, (ii) the centre of this conic, (iii) one point on this conic, other than P or Q.

6. A parabola has its axis parallel to $y=0$, passes through the origin, and touches the line $2y-x=3$ at the point $P(1, 2)$. Determine the directions of the asymptotes of the rectangular hyperbola which has four-point contact with the parabola at P.

7. Construct the circle outpolar to the hyperbola

$$x^2-2y^2=4$$

and passing through the points $(1, 4)$, $(3, 0)$.

Find the other points of contact with the hyperbola of the triangle, self-polar for the circle, of which one side is the tangent to the hyperbola at $(2\sqrt{2}, \sqrt{2})$.

8. A conic is outpolar to the circle

$$x^2+y^2=9$$

and passes through the points $(4, 1)$, $(0, 5)$, $(2, 2)$ and $(6, 3)$.

Construct (i) a fifth point on it, (ii) the tangents at this point and at the four given points, (iii) the centre.

CHAPTER XIV

THE CONE AND SPHERE

236. The geometry of the star. The lines and planes through a point O form a set of elements, which we shall term a **star** (*cf.* Arts. 1, 134, 135), following the modern practice, though the name **sheaf** is still often used. The lines and planes of the star meet any plane not passing through O in the points and lines of the plane respectively. To every geometrical theorem concerning points and lines of the plane there is a corresponding theorem concerning lines and planes of the star.

A range of the first order in the plane corresponds to a flat pencil in the star; a flat pencil in the plane corresponds to an axial pencil in the star, of which the axis is the line joining the vertex of the star to the vertex of the flat pencil.

We note further that, if a point of the plane lies on a line of the plane, the corresponding line of the star lies in the corresponding plane of the star. Thus properties of incidence are preserved when we pass from the plane to the star.

Since flat and axial pencils of the star are incident respectively with ranges and flat pencils of the plane, and incident forms are equi-anharmonic, it follows that properties involving cross-ratio are preserved when we pass from the plane to the star.

Further the correspondence between the lines of the star and the points of the plane, as also between the planes of the star and the lines of the plane is one-one and algebraic in the sense explained in Art. 158.

To the points of a plane curve correspond the generators of a cone of vertex O, standing upon the curve as base. To the tangents to this curve correspond tangent planes to the cone, their points of contact corresponding to the generators of contact of the corresponding tangent planes to the cone. Thus properties of tangency are preserved in the passage from the plane to the star.

It follows from the above that all projective properties of plane figures, that is, properties of incidence, tangency and cross-ratio, lead to corresponding properties of the star. In what follows we shall enumerate the most important of these properties. The proofs will in general be obvious from the above principles; where they are not so obvious, a few hints will be given to enable the student to supply the demonstration for himself.

As, however, a star cannot be projected into another star, as a plane figure can be projected into another plane figure, it is hardly legitimate to use the term projective of star figures, and we shall therefore use the more general word homographic in this connection.

The figure in the star which corresponds to a triangle in the plane is a trihedral angle, that is, a solid angle with three plane faces meeting at a point. This we shall call a **three-edge.** The plane faces correspond to the sides of a plane triangle; their intersections are the three edges, which correspond to the vertices of the plane triangle. The angles between the edges are the plane angles of the three-edge, and are analogous to the lengths of the sides in a plane triangle. The dihedral angles between the plane faces are analogous to the angles of the plane triangle.

Corresponding to a complete quadrangle in the plane we have a complete **four-edge** in the sheaf. Such a four-edge has six faces and the meets of the pairs of opposite faces form its diagonal three-edge.

Similarly to the complete quadrilateral corresponds the complete **four-face**, with six edges and three diagonal planes, which form its diagonal three-edge. The harmonic properties of the complete quadrangle and quadrilateral are transferred at once to the four-edge and four-face. Thus two faces of the diagonal three-edge of a complete four-edge are harmonically conjugate with regard to the two faces of the four-edge through their intersections; and two edges of the diagonal three-edge of a complete four-face are harmonically conjugate with regard to the two edges of the four-face in their common plane.

The reader should note that flat pencils in the star, although they have a common vertex, are not in general coplanar and therefore are not cobasal (Art. 24). Two such non-cobasal flat pencils are analogous, in the star, to two ranges on different straight lines in the plane.

We note also that the properties of two projective ranges or pencils, given in Chapter III, transfer at once to the star. Thus if two homographic flat pencils with a common vertex, but in different planes, have a self-corresponding ray, then corresponding rays lie in planes through a fixed axis, so that the two flat pencils are sections of the same axial pencil. Again if two homographic axial pencils whose axes intersect at O have a self-corresponding plane, corresponding planes meet on a fixed plane through O, so that the axial pencils are incident with the same flat pencil.

Also any two homographic non-cobasal axial pencils $[\alpha]$, $[\alpha']$ of the star have a **cross-axis** through which passes the plane joining the cross-meets $\alpha_1\alpha_2'$, $\alpha_1'\alpha_2$ of any two corresponding pairs of planes α_1, α_2 and α_1', α_2'. Similarly two homographic non-coplanar flat pencils of the star have a **cross-plane,** on which the planes a_1a_2', $a_1'a_2$ meet, where a_1, a_2 and a_1', a_2' are any two corresponding pairs of rays of the flat pencils.

237. Star perspective. Homographic and reciprocal star-fields. If we consider two coplanar fields in plane perspective, and form the corresponding star of vertex O, we obtain two **star-fields** homographically related in such a way that the plane through any two corresponding lines passes through a fixed line, and the meets of corresponding planes lie in a fixed plane. We may describe such a relation as **star perspective,** the fixed line being the **axis of star perspective** and the fixed plane the **plane of perspective.**

As in Chapter I, a star perspective is defined if we are given the axis and plane of perspective and either a pair of corresponding lines, or a pair of corresponding planes.

The property of Desargues' perspective triangles is immediately applied to the star. If abc, $a'b'c'$ be two three-edges of the star, whose faces are α, β, γ; α', β', γ' respectively, then, if aa' bb', cc' are concurrent through a line x, then $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ are coplanar in a plane π; and conversely.

We can also have the more general case where two star-fields are homographically related by a one-one algebraic relation, in which lines correspond to lines and planes to planes. Two such star-fields meet any plane in homographic plane-fields. The correspondence is uniquely determined when two corresponding four-edges are given.

In a similar manner, reciprocal fields in the plane give rise to reciprocal star-fields, having analogous properties. The principle of duality also applies to the star, lines and planes being now interchangeable.

Examples

1. Prove that the homography between two star-fields is entirely determined when two corresponding four-edges are given.

If the star-fields have the same vertex, show that they have, in general, one self-corresponding three-edge and that this three-edge, together with a pair of corresponding points, determines entirely the homography.

2. Prove that if two star-fields are reciprocal, these are two cones which are the locus and the envelope, respectively, of incident corresponding lines and planes.

3. Prove that the transformation, in which any line corresponds to the plane of the star perpendicular to the given line, is a reciprocal transformation in the star, and show that it corresponds to a point reciprocation in the plane, the radius of reciprocation being a pure imaginary.

238. Representation of the star on a sphere. If we describe a sphere of arbitrary radius, whose centre is the vertex of a star, every plane of the star determines a great circle on the sphere. Also every line of the star determines a pair of antipodal points on the sphere. From many points of view it is desirable to associate two such antipodal points as one unit, and we shall refer to them as a **dyad,** which may be denoted by either of its points. Two distinct great circles have only one dyad in common, and two distinct dyads determine a single great circle. Corresponding to any figure of points and lines in a plane there is a figure of lines and planes in the star, giving rise to a corresponding figure of dyads and great circles on the sphere.

An axial pencil of planes of the star determines a **spherical pencil** of great circles passing through a dyad and a flat pencil of lines of the star determines a **spherical range** of dyads on a great circle. The cross-ratio of four elements is determined from four arcs of a spherical range, or from four angles of a spherical pencil, by a formula involving the sines of these arcs or angles, identical with that proved for flat pencils in Art. 22. It is easily seen that the cross-ratio of four dyads on a great circle is independent of which particular point of any one dyad is chosen to fix the arcs in question. Also all great circles meet a spherical pencil of four great circles in spherical ranges of the same cross-ratio.

We have thus a whole theory of projective (*i.e.* homographic) and perspective forms of the first order on the sphere which corresponds to the theory already developed for the plane. There are certain differences, for example, bearing in mind that two points of a spherical range correspond to one line of the defining flat pencil through the centre of the sphere we see that there are *two* points (but only *one* dyad) of a spherical range determining a given cross-ratio with three given points of the range.

The reader will notice that dyads on a great circle form an involution on the circle, so that homographic spherical ranges are a particular case of homographic involutions.

Also the principle of duality will hold for figures on the sphere. For since the angles between two great circles are equal to the arcs joining their poles (measured by the angles subtended at the centre), if we make a great circle correspond to its pair of poles and conversely, we have spherical pencils corresponding to equi-anharmonic spherical ranges and conversely.

In order to avoid confusion with pole and polar with respect to a conic (which, as we shall see, has its analogue on the sphere) we shall speak of the poles of a great circle as its **spherical poles ;** the great circle will be spoken of as the **equator** of either of these points.

A three-edge of the star corresponds to a set of three dyads, forming a spherical triangle and its complements ; the arcs which form the sides of the spherical triangle measure the plane angles of the three-edge, and the angles of the spherical triangle are the dihedral angles of the three-edge.

The four-edge and four-face lead to a spherical quadrangle of dyads and to a spherical quadrilateral respectively. These have clearly the same harmonic property as the plane quadrangle and quadrilateral.

We thus have a further correspondence between the star and the sphere, besides that between the star and the plane. The correspondence between the star and the sphere is, however, more complete and intimate, for it preserves the symmetry round the centre. In fact, the difference between the geometry of the star and that of the sphere is merely one of language. It is, however, very convenient to use the sphere to represent properties of the star, because it enables us to use the same language as that of plane geometry, the great circle replacing the straight line, and the analogies between the geometry of the star and that of the plane are thus brought out in a striking manner.

It follows that, so far as *purely projective* properties are concerned, the spherical and the plane geometry must be entirely identical. For example, the theorems of the cross-centre and cross-axis, proved in Chapter II for projective ranges and pencils in the plane, apply equally to homographic great circle ranges and pencils on the sphere, and so do the constructions for corresponding triads, and for harmonic points and lines.

It is only when so-called *metrical* properties begin to come in, that is, the magnitudes of lines and angles, that the analogy breaks down. These properties, in the plane, invariably depend on the introduction of a special line and points, namely, the line at infinity and the circular points at infinity. These have no direct analogues on the sphere; and it is for this reason that corresponding metrical theorems in the two geometries are often so widely different.

We may note that the methods of plane perspective are immediately applied to the sphere, for star perspective leads at once to a **spherical perspective,** in which the great circle join of two corresponding points passes through a fixed dyad (the **pole of spherical perspective**) and the intersection of corresponding great circles lies on a fixed great circle (**the axis of spherical perspective**).

As in the plane, points on the axis, and great circles through the pole, are self-corresponding. There are, however, no great circles corresponding to the vanishing lines, since no great circle is the analogue of the line at infinity.

If two figures in plane perspective are joined to a point O outside their plane by lines and planes, and the whole cut by any sphere centre O, we obtain two figures in spherical perspective on the sphere.

Similarly two homographic plane fields project from a point O outside both their planes into two homographic star-fields with a common vertex; from which we obtain, on the sphere centre O, two **homographic spherical fields.**

EXAMPLES

1. Three great circles of a sphere through V are met by a transversal at A, B, C respectively. Prove that $\sin AB : \sin AC = \sin VB . \sin AVB : \sin VC \sin AVC$.

2. Prove that in two homographic fields on the sphere, the ratio

$$\frac{\sin BL}{\sin LC} \cdot \frac{\sin CM}{\sin MA} \cdot \frac{\sin AN}{\sin NB},$$

where L, M, N are any points on the sides BC, CA, AB of a spherical triangle, is the same for corresponding figures in the two fields.

3. State and prove the theorems corresponding for the sphere to the theorems of Ceva and Menelaus for the plane.

239. The cone of the second order and the sphero-conic. A cone of second order is defined as one which is cut in two points only by any straight line u in space. Joining u to the vertex O of the cone we see that the cone is cut by any plane through its vertex in two lines of the star through O. Hence its intersection with any plane is a curve of the second degree or conic.

Conversely to a conic in the plane corresponds a cone of the second order in the star.

The twin curve in which a cone of the second order meets a concentric sphere is called a **sphero-conic.** The properties of sphero-conics are merely a restatement in suitable language of the properties of the cone of second order. The student will find it a useful exercise, as he proceeds, to tabulate, in parallel columns, corresponding properties of the plane conic, the cone of the second order and the sphero-conic.

Now two coplanar conics s, s' can always be derived one from the other by a plane perspective. If the conics touch, this has been shown in Arts. 41–43. If they do not touch, let two of their common tangents a, b, whose points of contact with s, s' are A, A' and B, B' respectively, meet at O, and let any other line through O meet s and s' at C and C' respectively (each of these being arbitrarily selected from two intersections). Then the triangles ABC, $A'B'C'$ define a plane perspective, in which the conic s $(AABBC)$ corresponds to the conic $(A'A'B'B'C')$, *i.e.* to s'.

Accordingly **any two cones of the second order with the same vertex can be brought into star perspective, that is, any cone of the second order can be derived in this way from a right circular cone.**

In like manner, any two sphero-conics on the same sphere may be related by a spherical perspective, and any sphero-conic may be derived from the trace upon the sphere of a right circular cone whose vertex is at the centre of the sphere, that is, from a small circle of the sphere, such a small circle playing a part in the theory closely (though, for the reasons already explained, not completely) analogous to that played by the circle in the plane theory.

Since the line at infinity in the plane corresponds to a plane of the star, and to a great circle of the sphere, which have no special significance, it follows that the two regions, into which the inside of a hyperbola is divided in the plane, cease to be separated in the star. Thus, all the lines of the star, which belong to the inside of the cone, form a single continuum, and the boundary of the cone encloses it entirely. But although the *lines* form a continuum, the points (in virtue of the central symmetry of a cone) fall into two vertically opposite and symmetrical half-cones, connected at the vertex. Thus all cones of the second order have the same topological characteristics.

Consequently there is only one type of sphero-conic, consisting of two antipodally situated ovals, the intersections of the sphere by the opposite halves of the cone. The distinction between the three types of plane conic, ellipse, parabola and hyperbola, is not reproduced on the sphere, since there is no special great circle corresponding to the line at infinity.

240. Projective properties of the cone of second order. We may enumerate a few of the purely projective properties of the cone of the second order, and of the sphero-conic, which follow immediately from those of the conic.

In the case of the sphero-conic, no change of wording is even required, the theorems applying unchanged, with the one alteration that, where straight line is used in the case of the conic, great circle should be read in the case of the sphero-conic and also that we remember that a point in the plane really corresponds to a dyad on the sphere.

For the cone of the second order we have the following :

Chasles' Theorem gives : if α be a fixed tangent plane to a cone of vertex O, x a fixed generator, π a variable tangent plane, p its generator of contact, the flat pencil $\alpha[\pi]$ is homographic with the axial pencil $x[p]$.

We deduce, as in Chapter III, that a variable tangent plane π cuts four fixed tangent planes in a flat pencil of constant cross-ratio ; or taking any tangent line t lying in π, t cuts these four tangent planes in a range of constant cross-ratio.

Also a variable generator to the cone determines with four fixed generators an axial pencil of constant cross-ratio.

Conversely the product of two homographic axial pencils whose axes x_1, x_2 intersect at O is a cone of the second order vertex O, having x_1, x_2 for a pair of generators, and the envelope of the planes determined by the corresponding rays of two homographic flat pencils having a common vertex but not lying in the same plane is again such a cone.

If the two homographic axial pencils have a self-corresponding plane, their axes intersect in this plane. The flat pencils in which they intersect any plane are perspective and the product of the two homographic axial pencils is a plane, together with the self-corresponding plane.

Pascal's Theorem gives: if a six-edged solid angle be inscribed in a cone of second order, the lines of intersection of opposite faces are coplanar.

Brianchon's Theorem gives: if a six-faced solid angle be circumscribed to a cone of the second order, the planes joining opposite edges pass through a line. By taking arbitrary points on the six edges of this solid angle and joining them we obtain the somewhat different enunciation: if a skew hexagon be circumscribed to a cone of the second order, the three diagonals joining opposite vertices intersect a line through the vertex of the cone.

241. Pole and polar properties of the cone of second order. To any line p through the vertex of a cone of second order corresponds a plane π through the vertex which is called the **diametral plane** of the cone conjugate to the **diameter** p. This is obtained by joining to the vertex of the cone the polar of the point in which p cuts any plane α with regard to the section of the cone by α. Also since cross-ratio is unaltered by projection, if P be any point of p and a line through P meet the cone at Q, R and π at P' then P, P' are harmonically conjugate with regard to Q, R. π is therefore also called the polar plane of P, and conversely P is a pole of π. We see that any plane through the vertex has an infinite number of poles, which all lie on its conjugate diameter.

If P' lies on the polar plane of P, conversely the polar plane of P' passes through P.

Such planes are called conjugate diametral planes. They meet any plane in two lines which are conjugate with regard to the section of the cone by that plane. From the property that two such conjugate lines are harmonically conjugate with regard to the two tangents from their intersection, we see that two conjugate diametral planes for the cone are harmonically conjugate with regard to the two tangent planes through their intersection.

Similarly conjugate diameters of the cone are harmonically conjugate with regard to the two generators of the cone in their plane.

The polar plane of the vertex is indeterminate: conversely every plane not passing through the vertex has the vertex for pole.

To a triangle self-polar with regard to any plane section of a cone of second order corresponds a trihedral angle or **three-edge** self-polar with regard to the cone, the edges of which pass through the vertices of the triangle. These edges form a set of three diameters conjugate pair and pair, and such that each is conjugate to the opposite face of the three-edge.

Chords parallel to any diameter of the cone (*i.e.* any line through the vertex) are bisected by the conjugate diametral plane. This plane also contains the generators of contact of the two tangent planes to the cone through the diameter in question.

A diameter of the cone passes through the centres of the sections of the cone by planes parallel to its conjugate diametral plane.

For let u, v, w be a self-polar three-edge, it will meet a plane parallel to vw in the vertices U, V^∞, W^∞ of a self-polar triangle for the section. U is thus the pole with regard to this section of $V^\infty W^\infty$, that is, of the line at infinity in the plane of the section. U is therefore the centre of the section.

Using the representation on a sphere, previously explained (Art. 238), it follows that all the usual pole and polar properties of the conic hold also for the sphero-conic, the line joining the centre of the sphere to a point P being conjugate, for the cone corresponding to the sphero-conic, to the diametral plane through the polar great circle of P with respect to the sphero-conic. The property of diameters bisecting conjugate chords, and also of the tangent at P being parallel to the diameter conjugate to that through P depend on the line at infinity and are not transferable to the sphere, where, indeed, there are no analogues to parallel lines.

The generators of a cone of the second order form a **conical pencil of the second order,** and four rays of this pencil have a cross-ratio defined by the cross-ratio of the axial pencil formed by the planes through the rays and any given generator of the cone.

Such conical pencils have properties corresponding strictly to those of ranges of the second order on the conic. In particular, two such homographic pencils on the same cone have a cross-plane, which meets the cone in the self-corresponding rays of the pencils.

The involution properties, so far as they do not involve the special elements, are also transferable to the star. Thus, an involution of rays on a cone of the second order has an axis of involution, through which passes the plane containing any pair of mates, and an involution plane, containing meets of planes $(pq, p'q')$ or $(pq', p'q)$, where p, p' ; q, q', are any two pairs of mates.

Similarly the set of planes tangent to a cone of the second order have properties analogous to those of the pencil of second order in the plane. We may apply, to such a set of tangent planes, the name of **wrap of the second order.**

It would be tedious to multiply such examples ; once the general principle has been grasped, no difficulty will be found in transferring to the star any property which does not depend upon the special elements.

EXAMPLES

1. a, b are two fixed lines of a star, l is a variable line of the star such that the planes al, bl meet a fixed plane of the star in lines p, q, which make a constant angle with one another. Show that l describes a cone of the second order.

2. Find the envelope of a plane of a star which moves so that its traces on two fixed planes of the star subtend a fixed dihedral angle at a fixed line of the star.

3. Prove the analogue of Carnot's Theorem for the sphere, namely that, if the sides of a spherical triangle ABC meet a sphero-conic at non-antipodal points P, P' ; Q, Q' ; R, R', then

$$\frac{\sin BP.\sin BP'}{\sin CP.\sin CP'}\cdot\frac{\sin CQ.\sin CQ'}{\sin AQ.\sin AQ'}\cdot\frac{\sin AR.\sin AR'}{\sin BR.\sin BR'}=1$$

4. Investigate an analogue on the sphere and in the star of Newton's Theorem on parallel chords of a conic.

242. Outpolar and inpolar cones. If a cone κ_1 of the second order be circumscribed about a three-edge self-polar for another such cone κ, it contains an infinity of such three-edges, and is said to be outpolar to κ. Similarly if a cone κ_2 be inscribed in a three-edge self-polar for κ, it is inscribed in an infinity of such three-edges and is said to be inpolar to κ.

If κ_1 is outpolar to κ, then κ is inpolar to κ_1.

If we take two three-edges self-polar for a cone κ, there exist a cone κ_1, circumscribed about both three-edges, and therefore outpolar to κ, and a cone κ_2, inscribed in both three-edges, and therefore inpolar to κ. κ_1 is then trihedrally circumscribed to κ_2, and any number of three-edges can be described, each of which is inscribed in κ_1 and circumscribed about κ_2.

Conversely, if κ_1 is trihedrally circumscribed to κ_2, then a cone κ exists which is inpolar to κ_1 and outpolar to κ_2.

In a precisely similar manner we have sphero-conics outpolar, inpolar and triangularly circumscribed to other sphero-conics.

243. The circle at infinity and the spherical cone. We now proceed to consider special curves and surfaces, which play in three-dimensional space a part similar to that played by the circular points and lines in the plane, and which will enable us to obtain metrical results in the geometry of the star and sphere. We have already mentioned (Art. 4) the plane at infinity which we will denote by τ^∞. This plane cuts any sphere σ in an imaginary circle, which it will be convenient to denote by $\odot$.

Now consider any plane π. This meets σ in a circle c, which contains the circular points at infinity of π, I^∞ and J^∞. I^∞ and J^∞ therefore lie on σ and on τ^∞, that is, they lie on $\odot$; $\odot$ is therefore the locus of the circular points at infinity in all planes. Conversely, if I^∞, J^∞ are any two points of $\odot$, a plane π through $I^\infty J^\infty$ meets σ in a circle c and I^∞, J^∞ are the circular points in that plane; it follows, incidentally, that parallel planes have the same circular points at infinity.

It now appears that the circle $\odot$ is a locus independent of the choice of the sphere σ, so that it is the common intersection of all spheres with the plane at infinity.

But, in any plane, every circle can be obtained as the intersection of the plane with some sphere; applying this to the plane at infinity, we see that $\odot$ is the only circle in this plane. It is therefore known as **the circle at infinity.**

If we join any point O to the points of the circle at infinity, we obtain a cone, which is termed the **spherical cone** through O. Every plane π meets a spherical cone in a circle. For take the line at infinity of π. It meets the cone on the circle at infinity. The two circular points at infinity of π are therefore on the section: hence the latter must be a circle.

Such a cone, being a surface of the second order passing through the circle at infinity, is to be also considered as a sphere. It is, in fact, a **point-sphere** and is the limiting case of a sphere of vanishingly small radius—precisely as a pair of circular lines form a point-circle. Hence the name spherical cone.

Since the two circular points at infinity in any plane are conjugate imaginary, it follows that the conjugate imaginary point to any point of $\odot$ is itself a point of $\odot$, so that $\odot$ is its own conjugate imaginary locus.

The reader may ask why this does not make it a real circle, in the same way that a straight line which is its own conjugate imaginary can be shown to be always real. The answer is that the circle at infinity is indeed determined by two *real* equations, namely that of any sphere and that of the plane at infinity. But the locus determined by such real equations need not itself be real, unless the equations are both linear, which is the case for the straight line.

Similarly the spherical cone with a real vertex is its own conjugate imaginary. Here there is only one equation, which is real, being of the form $x^2+y^2+z^2=0$, but which has no real non-zero solutions.

244. Rectangular directions are conjugate for the circle at infinity. Let U^∞, V^∞ be two points at infinity corresponding to perpendicular directions. Let π be any plane through $U^\infty V^\infty$, meeting the circle at infinity at I^∞, J^∞. Then I^∞, J^∞ are circular points of π. Also, if P be any point of π at a finite distance, PU^∞, PV^∞ are perpendicular, and therefore harmonically conjugate with respect to PI^∞, PJ^∞. Hence U^∞, V^∞ are harmonically conjugate with respect to the points I^∞, J^∞ at which the line at infinity of π, that is $U^\infty V^\infty$, meets $\odot$. Hence U^∞, V^∞ are conjugate points for $\odot$.

Conversely, if U^∞, V^∞ are conjugate points for $\odot$, let $U^\infty V^\infty$ meet $\odot$ at I^∞, J^∞, then $\{U^\infty I^\infty V^\infty J^\infty\} = -1$. If π be any plane through U^∞, V^∞, then I^∞ and J^∞ are circular points of π and, P having the same meaning as before, PU^∞, PV^∞ are harmonically conjugate with respect to the circular lines through P, and therefore perpendicular. Thus U^∞, V^∞ correspond to rectangular directions.

Transferring the above results from the plane at infinity to the star of vertex O, we see that any two rectangular lines through O are conjugate for the spherical cone through O, and, conversely, that any two lines through O conjugate for this spherical cone are rectangular.

It follows that **any self-polar three-edge for the spherical cone is trirectangular,** which involves that its three faces are mutually perpendicular.

Since any two conjugate diametral planes can always be taken to form two faces of a self-polar three-edge, it follows that any two conjugate diametral planes of the spherical cone are perpendicular.

Transferring back the last property from the star to the plane at infinity we see that any two lines at infinity, conjugate for the circle at infinity, lie in perpendicular planes.

Conversely, **any rectangular three-edge is self-polar for the spherical cone through its vertex, and any two perpendicular planes through this vertex are conjugate for the spherical cone, and meet the plane at infinity in lines conjugate for $\odot$.**

Since the circle at infinity lies on any sphere, it is a particular case of a sphero-conic, and as such, plays a fundamental part in the metrical geometry of the sphere. It follows at once from the above that **any two points of the sphere, conjugate for ⊙, are separated by a quadrantal arc, that any two perpendicular great circles are conjugate for ⊙, and that any trirectangular spherical triangle is self-polar for ⊙**; the converse results being also true.

EXAMPLES

1. Show that if one rectangular three-edge can be inscribed in or circumscribed to a cone of the second order, an infinite number of such three-edges can be so inscribed or circumscribed.

2. Show that on a sphere the points of contact of great circle arcs through P which touch the circle at infinity are the intersections of this circle with the equator of P.

3. Prove that the tangent planes through any straight line in space to the circle at infinity touch this circle at the circular points in the plane perpendicular to the given line.

4. If C be any origin on a sphere, CX and CY two perpendicular great circles through C, M and N the feet of the great circle perpendiculars from any point P on CX, CY respectively, prove that the equation to any small circle, centre C and angular radius CA is given by

$$\tan^2 CM + \tan^2 CN = \tan^2 CA.$$

Hence, or otherwise, prove that the equation to the circle at infinity, in these co-ordinates, is

$$\tan^2 CM + \tan^2 CN = -1.$$

5. Prove that, if two great circle arcs OA, OB on the sphere meet at an angle θ, and OI, OJ are the two tangents from O to the circle at infinity then $O\{AIBJ\} = e^{2i\theta}$.

245. Principal axes and principal diametral planes of a cone of the second order. Since two conics have in general one common self-polar triangle, by considering the two cones having the same vertex and standing on these conics as bases, we see that two cones with a common vertex have, in general, one common self-polar three-edge. Taking one of the cones to be spherical we see that any cone of the second order has one rectangular self-polar three-edge. The faces of this three-edge are then clearly planes of symmetry for the cone, since each bisects chords of the cone perpendicular to itself, and any one of them cuts a section made by a plane parallel to another in an axis of this section.

A degenerate case arises when the given cone and the spherical cone have simple contact along a generator. In this case the common self-polar three-edge is degenerate, two of its faces coinciding with the common tangent plane, which is then to be regarded as perpendicular to itself. This case cannot, however, occur if the given cone of the second order is real, for, if it touches the spherical cone along an imaginary generator, it must also touch it along the conjugate imaginary generator, so that the cones have double contact.

But if two such cones have double contact, the conics in which they intersect any plane have also double contact, and have an infinity of common self-polar triangles with a common vertex. The cones have therefore an infinity of common self-polar three-edges with a common edge.

We deduce that if a cone has double contact with the spherical cone having the same vertex, it has an infinity of rectangular self-polar three-edges with a common edge and therefore (because trirectangular) with their other edges all coplanar. A plane perpendicular to this common edge will meet the given cone in a conic for which all pairs of conjugate diameters are perpendicular, that is, in a circle, through the centre of which the common edge in question passes. The cone is then a right circular cone, the common edge of the self-polar trirectangular three-edges being the axis of the cone.

A cone of the second order which has double contact with the spherical cone is therefore right circular, and conversely a right circular cone has double contact with the spherical cone.

Similarly, a sphero-conic has, in general, three mutually conjugate quadrantal dyads C_1, C_1'; C_2, C_2'; C_3, C_3', which are the intersections of three mutually conjugate perpendicular great circles. These latter may be spoken of as the three axes of the sphero-conic, and their intersections as the three (dyad) centres. If, through any centre C_1, a great circle arc be drawn, meeting the sphero-conic in two dyads (P, P') (Q, Q') and the axis C_2C_3, which is the polar of C_1, at R, then C_1, R are harmonically conjugate with regard to the dyads (P, P') (Q, Q'); but since C_1, R are separated by a quadrant, C_1, R bisect the angles between PP' and QQ'. Thus the points of the sphero-conic fall into pairs symmetrically situated with respect to C_1. The centres are thus points of symmetry. Similarly, P, P', Q, Q' fall into pairs symmetrical with respect to R. Since R is on an axis C_2C_3, and the great circle C_1R is perpendicular to this axis, the sphero-conic is symmetrical with respect to this (great circle) axis.

If the sphero-conic has double contact with the circle at infinity, it becomes a pair of antipodal circles. If we take the centres

C_1, C_1' of these as one dyad centre, their equator is an axis, but any two quadrantal points on this equator are possible centres, and any two perpendicular great circles through C_1, C_1' are possible axes. Thus the other centres and axes are indeterminate.

246. Foci of sphero-conic and focal lines of a cone. Any sphero-conic s has four common great circle tangents with the circle $\odot$ at infinity. These form a complete spherical quadrilateral with six dyad vertices. The twelve points of these are called the foci of the sphero-conic.

Clearly two great circles through a focus which are harmonically conjugate with regard to the common tangents through that focus are conjugate for both s and $\odot$, and therefore perpendicular. The foci are therefore points such that great circles conjugate for s, and passing through a focus, are perpendicular.

The foci lie in fours on the axes of the sphero-conic; for a pair of opposite dyad vertices lie on a diagonal of the spherical quadrilateral above mentioned, that is, on a side of the common self-polar triangle of s and $\odot$, and this is an axis of the sphero-conic.

The polars of the six focal dyads with respect to s are called the directrices of s.

Since we have seen that $\odot$ is its own conjugate imaginary, and the same is true of s if s is real, the four common tangents x, x', y, y' must be conjugate imaginary in pairs. Let x, x' be conjugate imaginary, as also y, y'. Then there can be only two real focal dyads (S, S') (H, H') given by xx' and yy' respectively. Since these are opposite vertices of the spherical quadrilateral $xx'yy'$, they lie on an axis, which may be called the focal axis of the sphero-conic. Of the other focal dyads, xy and $x'y'$ are conjugate imaginary, so that the arc joining them is real; similarly xy' and $x'y$ are conjugate imaginary, and the arc joining them is real. The three axes and the centre dyads are thus always real. The directrices are real only when the corresponding focal dyads are real.

Transferring these results to the star, we see that a cone of the second order has three real principal axes and three real principal diametral planes, on which lie in pairs six **focal lines,** but only one pair of these is real. The focal lines are the edges of the four-face formed by the common tangent planes of the given cone and the spherical cone. Each of them has the property that conjugate diametral planes through it are perpendicular.

Since the tangent planes from the real focal lines to the spherical cone are clearly imaginary, no real tangent planes can be drawn to the given cone of the second order through the real focal lines, and these must therefore lie entirely inside the cone. Hence the real foci of a sphero-conic lie in pairs inside the two antipodal ovals of which the curve consists.

Because a diagonal of a spherical quadrilateral is harmonically divided by the other two diagonals, the centre dyads (C_1, C_1') (C_2, C_2') in the focal axis of a sphero-conic are harmonically divided by the focal dyads (S, S') (H, H') and, being quadrantal, bisect the angles between these pairs. Hence the focal dyads are symmetrical with respect to the centre dyads.

Similarly in the cone, two focal lines in a principal diametral plane are equally inclined to the axes in that plane.

247. Focal properties of the sphero-conic. A number of focal properties of the conic are repeated in the sphero-conic.

If P be any point of the sphere, there are, in general, two arcs u, v through P which are conjugate for both $\odot$ and the sphero-conic s, being the common mates of the involutions of conjugate great circles through P for $\odot$ and s respectively. They are thus conjugate arcs at right angles. Since they are conjugate for $\odot$, s, they are also conjugate (by Art. 211, transferred to the sphero-conic) for all sphero-conics of the range defined by $\odot$, s; in particular for the dyad-pair defined by two focal dyads (S, S') (H, H') on the same axis. Accordingly PS, PH are harmonically divided by the perpendicular arcs u, v. These latter are accordingly the bisectors of the angle SPH.

Further, if tangent arcs t_1, t_2 be drawn from P to the sphero-conic, t_1 and t_2 also are harmonically conjugate with respect to u and v and so the angle between t_1, t_2 is bisected by u and v.

The same is true of the pair of tangents from P to any sphero-conic of the range $(\odot, s)$, all of which sphero-conics are confocal.

In particular, if P be on the conic s, t_1 and t_2 coincide with the tangent at P to s, so that u, v coincide with the tangent and normal. We have thus the result that the tangent and normal bisect the angle between the focal distances.

Proceeding on these lines we can prove the properties of Art. 218 for confocal sphero-conics.

Again, if the tangents t_1, t_2 from P to s touch s at T_1, T_2, T_1T_2 is the polar arc of P. If it meet the directrix corresponding to the focal dyad (S, S') in the dyad (Y, Y'), then SP is the polar of the dyad (Y, Y') and SY, SP are conjugate arcs for s and therefore perpendicular. But if Z is the meet of the arcs T_1T_2 and SP, T_1 and T_2 are harmonically conjugate with regard to Z and Y. Hence SP, SY are the bisectors of the angles between ST_1, ST_2, so that two tangents subtend equal or supplementary angles at a focus. It is easily shown that the angles are equal if T_1, T_2 are taken on the same oval of the sphero-conic, and supplementary if taken on different ovals ; the two cases, however, correspond here to the same tangents, and merely involve a selection between antipodal points.

Examples

1. Show that if a pair of tangent planes through a diameter d of a cone of second order touch the cone along s, t and f be a focal line, the planes fs, ft are equally inclined to fd.

2. If a tangent plane to a cone of the second order meet the tangent planes perpendicular to a principal diametral plane in lines x, y, the lines x, y subtend a right dihedral angle at a focal line situated in the given principal diametral plane.
State the corresponding theorem for the sphero-conic.

3. Show that a plane perpendicular to a focal line cuts the cone in a conic, the focal line in question passing through a focus of this conic.

4. Prove that, on an axis of a sphero-conic, any number of pairs of dyads (P, P') (Q, Q') can be found such that all conjugate great circles through (P, P') and (Q, Q') respectively intersect at right angles, and that these dyads form an involution, of which the focal dyads are double elements.

248. The focus and directrix property for the sphero-conic. Let O (Fig. 76) be a focus of a sphero-conic s, i its directrix, i' its equator, x any great circle passing through the intersections X, Y of i and i', and which may conveniently be taken as passing through O.

Then O as pole, x as axis of perspective, and i, i' as a pair of corresponding great circles define a spherical perspective. Let s' be the sphero-conic corresponding to s in this perspective. Since i is the polar of O for s, and O is self-corresponding, i' is the polar of O for s', and, since the arc from O to any point of i' is a quadrant, i' is also the polar of O for $\odot$. Therefore O and i' are a centre and axis of s' respectively. But, since conjugate arcs through O for s (which are perpendicular) correspond to themselves, they are also conjugate for s' ; hence all conjugate arcs through O for s' are perpendicular. This is the case where the axes of s' through O are indeterminate, so that s' reduces to a pair of antipodal small circles centre O.

Let the great circle orthogonal to i, i' (and which passes through O) meet i and i' at I, I' respectively. Let P be any point of s. Join the great circle IP, meeting x at Z, and join ZI' meeting the arc OP at P'. Note that in all these constructions each point may be either element of a dyad.

By the properties of spherical perspective P' is a point of s' corresponding to P. Let the arc OPP' meet i at J and i' at J'. If now we project (spherically) the range $OPJ'P'$ from I' upon

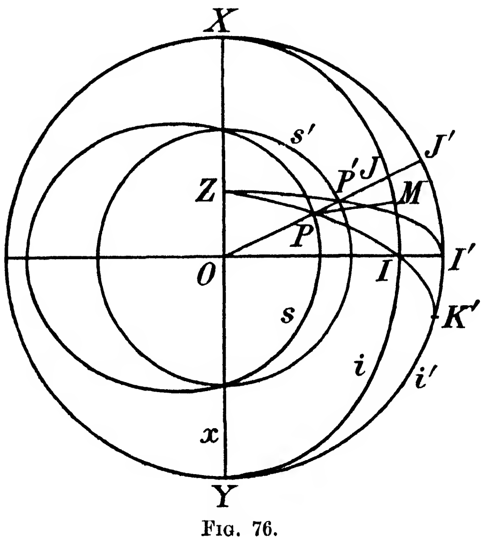


FIG. 76.

the great circle ZI, we obtain the range $IPK'Z$, if K' is the point where the circle ZI meets i'. We have then

$$\{OPJ'P'\} = \{IPK'Z\}.$$

Next project $IPK'Z$ from X back upon the great circle OP, we obtain the range $JPJ'O$, so that

$$\{IPK'Z\} = \{JPJ'O\}.$$

Combining these two results

$$\{OPJ'P'\} = \{JPJ'O\}.$$

Writing out the cross-ratios, this gives

$$\frac{\sin OP . \sin J'P'}{\sin OP' . \sin J'P} = \frac{\sin JP . \sin J'O}{\sin JO . \sin J'P}.$$

Cancelling out the common factor sin $J'P$ in the denominators, and remembering that $J'O$ is a quadrant and that OP' and $P'J'$ are complementary arcs, so that $\sin P'J' = \cos OP'$, we have

$$\frac{\sin OP}{\tan OP'} = \frac{\sin JP}{\sin JO}.$$

If now the arc PM is drawn perpendicular to the directrix i of s, meeting i at M, PJM and OJI are two right-angled spherical triangles, having a common angle at J. By a well-known result

$$\frac{\sin JP}{\sin JO} = \frac{\sin PM}{\sin OI}.$$

Hence

$$\frac{\sin OP}{\tan OP'} = \frac{\sin PM}{\sin OI}$$

or

$$\frac{\sin OP}{\sin PM} = \frac{\tan OP'}{\sin OI}.$$

Since s' is a pair of antipodal circles centre O, OP' is a constant arc. Also OI is clearly independent of the choice of P. Hence, in the last written equation, the right hand side is a constant, which we can denote by e, and which may be called the eccentricity of the sphero-conic.

We have thus

$$\sin OP = e.\sin PM,$$

or the sines of the distances from the focus and directrix respectively are to one another in the constant ratio of the eccentricity.

It is to be noticed, however, that the numerical value of the eccentricity is not necessarily, as in the plane, an indication of the shape of the curve. Thus, if the sphero-conic is a small circle O, the directrix is the equator of O, so that the arcs OP, PM make up a quadrant, and $\sin PM = \cos OP$, so that

$$e = \tan \text{(angular radius of circle)}.$$

The eccentricity of a circle is thus not zero on the sphere, unless the angular radius of the circle tends to zero.

Example

If S is a focus of a sphero conic, X the point where the focal axis meets the directrix corresponding to X, C a centre on the focal axis, A a vertex of the curve, lying between S and X, prove that

$$\tan CA.\cos CS = e \sin CX$$
$$\cot CA.\sin CS = e \cos CX$$

and hence that

$$\frac{\cos^2 CS}{\cos^2 CA} + \frac{\sin^2 CS}{\sin^2 CA} = 1 + e^2.$$

249. The sum and difference of the focal distances. To prove that, if P be any point of a sphero-conic, S, H two non-antipodal foci, then the sum or difference of the arcs SP, HP is constant.

Suppose, to begin with, that S and H are foci lying within the same oval of the sphero-conic (Fig. 77).

Let TP, TQ be two tangent great circles to the sphero-conic from T, the points of contact P and Q being taken on the oval belonging to S, H. Then the angle STP is equal to the angle HTQ,

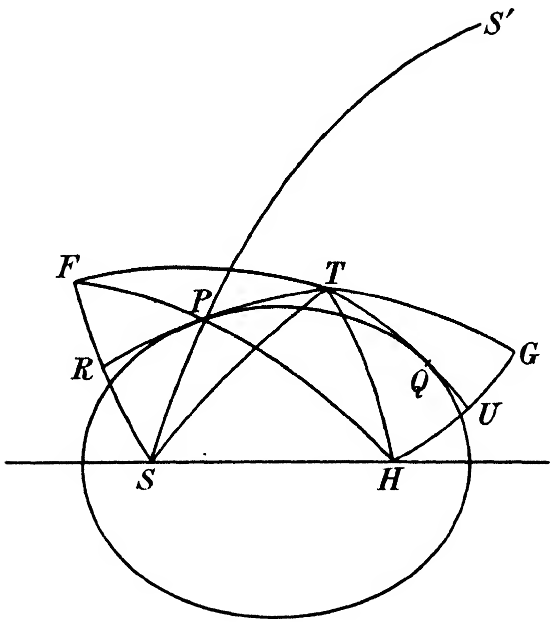


Fig. 77.

since, by Art. 247, the perpendicular conjugates through T bisect both angles STH, PTQ.

Let F, G be the points symmetrical to S, H with regard to TP, TQ respectively. Then the angles FTP, PTS, HTQ, QTG are all equal, and the angle FTS = angle HTG.

Adding angle STH we have angle FTH = angle STG, arc FT = arc ST and arc TH = arc TG.

The spherical triangles FTH, STG are congruent and arc FH = arc SG.

Now if the tangent great circle PT move round the curve, TQ

remaining fixed, G remains fixed and SG remains fixed, $\therefore$ arc FH = a constant length.

Joining PF, PS, PH, angle $FPR = SPR$ by symmetry; and $SPR = TPH$ (Art. 247). Hence angle $FPR = TPH$ or FPH is a great circle. Thus $SP + PH = FP + PH = FH$ = constant.

If we take two foci S', H not inside the same oval, then SP = semi-circumference $- S'P$. Thus $PH - S'P$ = constant.

We notice therefore that, from the point of view of focal distances, antipodal ovals of a sphero-conic behave as opposite branches of a hyperbola.

250. Cyclic planes of a cone and cyclic arcs of a sphero-conic. A sphero-conic s determines with the circle $\odot$ at infinity four common dyads (A_1, A_1') (A_2, A_2') (B_1, B_1') (B_2, B_2'), which if s is real fall into conjugate imaginary pairs (A_1, A_1'), (A_2, A_2') and (B_1, B_1'), (B_2, B_2'). The six great circles which join pairs of these four dyads, and which form the sides of the complete quadrangle determined by s and $\bigcirc$ are termed **cyclic lines** or arcs of the sphero-conic.

Clearly the intersections of opposite cyclic lines are dyad vertices of the common self-polar triangle of s and $\bigcirc$, that is, each pair of opposite cyclic lines pass through a centre-dyad of the sphero-conic.

Of the six cyclic lines, only two are real, namely those which join the conjugate imaginary pairs, that is, the great circles $A_1A_2A_1'A_2'$ and $B_1B_2B_1'B_2'$. The others are conjugate imaginary in pairs.

Any two dyads which are conjugate for both s and $\odot$, that is, any two quadrantal conjugate dyads of s, are conjugate for the great circle pairs formed by opposite cyclic arcs.

Obviously two such quadrantal dyads are any two non-antipodal centres, C_1, C_2, say. These must be conjugate for the great circle pair passing through C_3. Hence C_3C_1, C_3C_2 are harmonic with regard to the cyclic lines through C_3. Since C_3C_1, C_3C_2 are rectangular, the cyclic lines are equally inclined to them.

Since the cyclic lines all meet the sphero-conic in imaginary points, it is clear that the real cyclic lines do not pass through the dyad centres (C_1, C_1') interior to the ovals.

The reader will have no difficulty in proving that, in the case where the sphero-conic s is a small circle, that is, has double contact with $\odot$, four of the cyclic arcs coincide with the equator of the centre of s, which is real if s is real, and the remaining two are the common tangent great circles of $\odot$ and s, which are conjugate imaginary if s is real.

Considering similarly the six faces of the common four-edge of a cone of the second order and the spherical cone, we see that a cone of the second order has six cyclic planes, which pass in pairs through each of the three axes, each pair being equally inclined to the principal diametral planes through that axis. Only one such pair is real.

Since a cyclic plane contains two circular points at infinity on the given cone, a plane π parallel to a cyclic plane contains the same two circular points, which therefore lie on the section of the cone by π; this section is accordingly a circle. Thus **the cyclic planes are planes parallel to the planes of circular section.**

Let α be any plane through the vertex O of the cone, meeting the cone in two generators a, b, and a pair of cyclic planes in rays x, y. There are two rays u, v in α which are conjugate for the pencil of cones through the intersections of the given cone and the spherical cone. Hence they are rectangular and harmonically conjugate with respect to a, b. But they are likewise conjugate for the plane-pair formed by the two given cyclic planes, since these are a cone of the pencil. Therefore u, v are harmonically conjugate with respect to x, y, as well as with respect to a, b. Being rectangular they bisect the angles between x, y, and also between a, b. Hence the angles between a and x are equal to the angles between b and y.

Projecting this result upon the sphere, if a great circle meet the sphero-conic at A, B and a pair of opposite cyclic arcs at XY, the arcs AX, BY are either equal or supplementary. It will be noticed that this result is analogous to a property of the asymptotes of a hyperbola (Art. 67) so that the cyclic lines correspond, but only in this limited sense, to the asymptotes.

EXAMPLES XIV

1. Discuss the nature of the homography on the sphere in which the self-corresponding triangle is trirectangular, and the intersections of a great circle through one of its vertices with the circle at infinity are corresponding points.

Show how to construct, in this case, the great circle corresponding to any given great circle, and the dyad corresponding to any given dyad.

Prove also that a second pair of conjugate imaginary points on the circle at infinity correspond.

2. Show that properties of a spherical figure may be reciprocated by making

a great circle correspond to its spherical poles and conversely. Either figure may then be called the spherical polar figure of the other.

Prove that the spherical polar of a sphero-conic is a sphero-conic, the cyclic lines of the one reciprocating into the foci of the other.

Show that this corresponds to polar reciprocation with regard to the circle at infinity.

3. Through two fixed points of a sphere great circle arcs are drawn which intersect at right angles. Prove that their intersection lies on a sphero-conic passing through the two fixed points.

4. If a principal diametral plane of a cone of the second order meet the cone in a, a' and p be any generator of the cone, the planes pa, pa' meet either cyclic plane through the axis perpendicular to the given principal plane in two lines at right angles.

5. If one side of a spherical triangle move so that the area of the triangle remains constant, it envelops a sphero-conic of which the other two sides (which remain fixed) are the cyclic lines.

[Deduce from Art. 249 by the method of Ex. 2.]

6. Prove that any sphere which passes through a section of a cone of the second order made by a plane parallel to one cyclic plane meets the cone again in a plane section parallel to the opposite cyclic plane.

7. Through any ray of a star through O two cones of a confocal system having O for vertex can be described and these are orthogonal.

8. Prove that a quadrantal arc whose extremities move on a sphero-conic envelops another sphero-conic, coaxial with the first one.

[For this arc touches the harmonic envelope of the sphero-conic and $\odot$.]

9. Prove that the intersection of two perpendicular tangents to a sphero-conic is a sphero-conic coaxial with the given one.

[Reciprocate on the sphere the result of Ex. 8 with respect to the circle at infinity.]

10. C_1 is the internal centre, C_2 and C_3 the two external centres of a sphero-conic; C_1C_2, C_1C_3 meet the curve at non-antipodal points A_1, A_2; B_1, B_2 respectively. P is a point on the curve, and the arcs C_3P, C_2P meet C_1C_2, C_1C_3 respectively at M, N.

Prove the following formulæ:

$$\frac{\sin^2 C_1M}{\sin^2 C_1A_1}+\frac{\tan^2 PM}{\tan^2 C_1B_1}=1, \qquad \frac{\tan^2 PN}{\tan^2 C_1A_1}+\frac{\sin^2 C_1N}{\sin^2 C_1B_1}=1,$$

$$\frac{\tan^2 C_1M}{\tan^2 C_1A_1}+\frac{\tan^2 C_1N}{\tan^2 C_1B_1}=1, \qquad \frac{\sin^2 PN}{\sin^2 C_1A_1}+\frac{\sin^2 PM}{\sin^2 C_1B_1}=1.$$

[Apply Carnot's Theorem to the triangles C_1C_3M, C_1C_2N.]

11. If C is the internal centre of a sphero-conic, S a focus internal to the oval belonging to C, CA and CB the semi-axes of this oval, CA being the focal semi-axis, prove the following results:

$$\sin^2 CS \sec^2 CA = \tan^2 CA - \tan^2 CB$$

$$e^2 = \sec^2 CB - \tan^2 CB \cot^2 CA.$$

Hence prove that CA is always greater than CB, and that the eccentricity corresponding to the real foci is always real.

12. Prove that the real cyclic arcs pass through the external centre on the focal axis, and meet this axis at an angle α given by

$$\sin \alpha = \frac{\sin CB}{\sin CA}.$$

13. Prove that in the notation of Ex. 12

$$\alpha = \text{angle } CSB,$$

where S is a real focus.

14. If two non-coplanar conics touch one another a cone of the second order passes through both of them.

[Use Arts. 9, 43.]

15. Prove that two cones of the second order which touch one another along the line joining their two vertices intersect in a conic.

CHAPTER XV

PROJECTIVE METHODS IN THREE DIMENSIONS

251. Order, class, degree. The **order** of a surface is the number of points in which it is met by any straight line not lying in it.

The **class** of a surface is the number of tangent planes which may be drawn to it through any straight line not lying in it.

The **degree** of a skew or twisted curve (that is, a curve which does not lie in a plane) is the number of points in which it is met by any plane.

Note that this definition of degree coincides with the one previously adopted for a plane curve, for the intersections of the latter with any straight line in its plane may also be looked upon as intersections with another plane through this straight line.

Note also that a plane section of a surface of the nth order is a curve of the nth degree.

The following result will be assumed: three surfaces of order m, n, p intersect in mnp points, real or imaginary. This is evident from analytical considerations if we remember that the equation to a surface of order n is of the nth degree in the co-ordinates.

EXAMPLE

Show that no non-degenerate twisted curve can be of degree less than three, so that a curve of the second degree in space is necessarily a conic, unless it breaks up into two skew lines.

252. Ultimate intersections and joins. Precisely as the intersection of two near tangents a, t to a plane curve coincides in the limit with the point of contact of a, when t coincides with a, so the intersection of a plane π, which varies in a specified manner, with a fixed plane α, when π ultimately coincides with α, coincides in general in the limit with a definite line l of α, which depends on the law of variation of π, assumed here to depend on a single parameter. This line l will be spoken of as the **ultimate intersection** of π and α, which may be expressed shortly by the notation $\pi\alpha \rightarrow l$.

Again, if a variable point P approaches a line a in a specified manner, and ultimately coincides with a point A of a, the plan aP will in general approach a limiting position α, which will be spoken of as the **ultimate join** of P and a, with the notation $Pa{\to}\alpha$. In like manner a variable line x will determine an ultimate join $Ax{\to}\alpha$ with a point A.

Similarly, a variable line x, not itself lying in a plane π, which approaches in a specified manner the plane π and ultimately coincides with a line l lying in π, determines on π a point P, whose limiting position A, which necessarily lies on l, will be the ultimate intersection $x\pi{\to}A$.

In general, however, if a different plane π' be taken through l, then the ultimate intersection $x\pi'{\to}A'$ leads to a point A' of l, different from A. There is no definite ultimate intersection of x and l, which must then be regarded as coinciding without intersecting.

But if, whatever the choice of the plane π through l, the ultimate intersection $x\pi{\to}A$ is always the same, x and l will be said to **intersect ultimately,** and A will be their ultimate intersection.

In like manner, if P be any point of l, the plane Px will, in general, approach a limit α, which contains l. If this limit α is independent of the choice of P on l, the two lines l, x will be said to be **ultimately coplanar,** and the plane α is their ultimate join.

We will now prove that if a variable line x ultimately intersects a line l, then it is ultimately coplanar with l.

Take any sphere centre O, of unit radius; draw from O a radius parallel to a given direction. Then the point in which this radius meets the sphere gives a convenient representation of that direction; directions which are parallel to a plane are represented by points lying on a great circle, which also represents the plane.

Assign on l, x positive directions which ultimately coincide; this is done to avoid the ambiguity which would result from having a given line represented by a dyad.

For clearness we will denote, *on this occasion only*, representative points and great circles on the sphere (Fig. 78) by the letters which denote the corresponding lines and planes in space.

An arbitrary plane μ is taken through l, meeting x at M. A is the ultimate intersection of x and l, P any other point on l, $AM \equiv a$, $PM \equiv p$. Then, as x varies, p ultimately coincides with l. Assign on p the direction which ultimately coincides with that of l. Let λ be the plane through l which is parallel to both x and l; this is represented by the arc xl on the sphere. λ is clearly independent of the choice of both μ and P, and it will, in general, approach some limiting plane β through l. Let π, α be the planes Px, Ax, shown by the arcs px, ax in Fig. 78. These are independent of the choice of μ, and α is independent of the choice of both μ and P.

Clearly a, p, l lie in μ. Further, A is the limit of M, whatever μ may be $\therefore$ since P remains constant as x varies, AM/MP approaches zero in the limit, or, from the plane triangle AMP, $\sin pl/\sin al$ approaches zero.

But, since in the spherical triangles lpx, lax the sines of the sides are proportional to the sines of the opposite angles

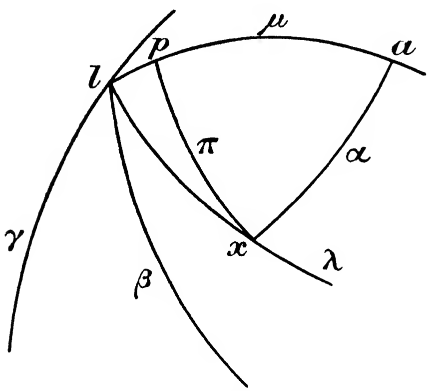


FIG. 78.

$$\frac{\sin pl}{\sin xl} = \frac{\sin \lambda\pi}{\sin \mu\pi},$$

$$\frac{\sin al}{\sin xl} = \frac{\sin \lambda\alpha}{\sin \mu\alpha}.$$

Hence, by division

$$\frac{\sin \lambda\pi}{\sin \lambda\alpha} \cdot \frac{\sin \mu\alpha}{\sin \mu\pi}$$

approaches zero.

Now α approaches the limit of the plane Ax; this is, in general, a definite plane γ through l. Thus the angle $\mu\alpha$ approaches $\mu\gamma$ and $\sin \mu\alpha$ approaches $\sin \mu\gamma$, which does not vanish if μ (which is arbitrary) is taken distinct from γ. And since $\sin \lambda\alpha$, $\sin \mu\pi$ are always less than unity, $\sin \mu\alpha/(\sin \lambda\alpha \sin \mu\pi)$ cannot approach zero, hence $\sin \lambda\pi$ must approach zero. Thus the plane π approaches parallelism with λ, and therefore with β. Moreover the limit of π must contain l, and so is the actual plane β. Hence Px tends to the limit β independently of the choice of P.

Since P may be taken as close as we please to A, considerations of continuity indicate that the limit of Ax must also, *in general*, be β, so that β and γ are the same. If this is not the case, the intersection A will be regarded as singular, and a special investigation is necessary.*

* It will be noticed that no use has been made, in the above proof, of the datum that the limit of $x\gamma$ is A. Thus in the definition of ultimate intersection of x, l, *one* plane γ might be exceptional. This would correspond to the

Reciprocating the above theorem we see that ultimately coplanar lines are also ultimately intersecting.

If a point P approaches a point A along a definite path p which passes through A, then the ultimate join AP is the tangent to p at A. If a second point Q also approaches A along a path q, then the ultimate join AQ is the tangent to q at A. The plane APQ will then approach a limiting position, namely the plane determined by the tangents to p and q at A. This is the ultimate join of the three points APQ. It may happen that the paths p and q coincide; in this case we may first make P coincide with A; the ultimate join then contains the tangent t at A to p and a neighbouring point Q, and the ultimate join Qt is the ultimate join of the three coincident points.

253. Curves and developables. If P is a given point on a twisted curve s, Q a variable point, then the ultimate join PQ, when Q coincides with P, is the tangent p to the curve at P. If R be any other point of the curve, then the ultimate join $PQR \to \pi$, or $pR \to \pi$, when Q and R coincide with P, is termed the **osculating plane** to the curve at P, and may be described as the plane through three coincident points of the curve at P.

Lines through P perpendicular to the tangent at P are termed **normals** to the curve; of these, the one which lies in the osculating plane is called the **principal normal,** and the one at right angles to the osculating plane the **binormal.** The plane through P perpendicular to the tangent is the **normal plane** at P. The plane containing the tangent and binormal is the **rectifying plane** at P.

If we apply to such a twisted curve s the principle of duality in space (Art. 135), we obtain the envelope σ' of a plane π' depending on a single variable parameter. This envelope is termed a **developable.** The ultimate intersection of π' with a neighbouring tangent plane which tends to coincide with π' is a line p' of π' which is the line of contact of π' with its envelope and so is a generator of σ'. Also π' is clearly the tangent plane to σ' at every point T' of p'. If now q' is a neighbouring generator, then, as q' approaches p', the lines joining T' to the points of q' become tangents to σ' in the limit and lie in the tangent plane π' at T', so that π' is the limiting position of the plane $T'q'$.

case where A, in the definition of ultimate join, is an exceptional point. If the definitions are modified in this sense, the proof given in the text is made general. It is better, however, not to do this, and to regard the ultimate intersection (or ultimate join) as of singular type in the exceptional cases.

Hence, by Art. 252, since π' is independent of the choice of T' on p', the generators p', q' ultimately intersect. If their ultimate intersection is P', then P' is (again by Art. 252) the ultimate intersection of q' with π', and, since q' is itself the ultimate intersection of tangent planes κ', ρ', therefore P' is the ultimate intersection of three coincident tangent planes π', κ', ρ' : this follows by reciprocating the statements in Art. 252 concerning the ultimate join of three coincident points. The locus s' of P' is termed the **cuspidal edge** of the developable σ'.

Returning now to the original curve s, where P corresponds to π', let Q, R correspond to κ', ρ' respectively. Thus P' corresponds to the ultimate join of three coincident points P, Q, R on s, that is, to the osculating plane π of s at P. The tangent p at P to s corresponds to the generator p' of the developable, and a neighbouring tangent q corresponds to the neighbouring generator q'. Since p', q' ultimately intersect, so do p and q, and we get the result that **tangents to a twisted curve determine an ultimate intersection and an ultimate join.**

Now the ultimate join of p', q' has been shown to be π' and their ultimate intersection P'. It follows that the ultimate intersection of p, q is the point of contact P of p, and their ultimate join is the osculating plane π at P.

But further, it is now clear that we may look upon a twisted curve s as being generated by the ultimate intersections of its system of tangents p. The correlative twisted curve s' is therefore generated by the set of corresponding lines p' as tangents. Thus the tangent at P' to the cuspidal edge of the developable σ' is the generator p' of the developable.

Now p' is the ultimate join of two points P' of the cuspidal edge in question, which approach coincidence. Hence p is the ultimate intersection of two osculating planes π of s, which approach coincidence. Accordingly **two neighbouring osculating planes at P ultimately intersect in the tangent at P.**

Moreover the planes π themselves envelop a developable, which is termed the **osculating developable** σ of the curve s. There is a complete reciprocity between the two systems, and the curve s is the cuspidal edge of its osculating developable.

The meaning of the term developable applied to such a type of surface may be illustrated as follows, but the argument is not to be taken as rigorous.

A succession of planes π_1, π_2, π_3, etc., selected in order, according to a law of variation involving a single parameter, determine a succession of intersections $\pi_1\pi_2 = g_{12}$, $\pi_2\pi_3 = g_{23}$, etc. Successive lines such as g_{12}, g_{23} meet at a point $\pi_1\pi_2\pi_3 = P_{123}$, and so on. The set of lines g are edges or creases of a polyhedral surface; this surface may be unfolded about the successive lines g and so spread out or "developed" upon a plane, the skew polygon formed by the g's becoming a plane polygon.

If now the number of planes π is indefinitely increased, successive planes being taken closer and closer together, we approach, in the limit, a continuous distribution, the g's tending to ultimate intersections, the points P to points of the cuspidal edge, which then becomes the limit of the skew polygon, and the polyhedral surface to a developable, which can still be unfolded into a plane.

The **class of a developable** is defined to be the number of tangent planes to the developable passing through a general point of space.

The **class of a twisted curve** is the number of osculating planes which pass through a general point. It is clearly identical with the class of its osculating developable. It should be noted that the order of a developable (the number of points in which it is met by any straight line) is necessarily equal to the number of its generators which intersect any straight line. If we define the order of a twisted curve as the number of tangents which meet any straight line, the order of a developable will be the same as the order of its cuspidal edge, though different, in general, from the *degree* of the latter.

The normal planes to a twisted curve s form a set of planes depending a single parameter (which might be the arc of the curve). These planes therefore envelop a developable, which is termed the **polar developable** of s. The intersection of the normal planes at P and Q is at right angles to the tangents p, q at P and Q. Proceeding to the limit, since p, q are ultimately coplanar, the ultimate intersection of the normal planes when Q approaches P is perpendicular to the ultimate join of p, q, that is, to the osculating plane at P. Any generator of the polar developable is therefore parallel to the binormal at the corresponding point of the curve.

Similarly the rectifying planes at points of a curve envelop a developable, termed the **rectifying developable.** The generators of the rectifying developable are the ultimate intersections of the rectifying planes; the generator corresponding to a point P of the curve is said to be the rectifying line at P.

254. Radii of curvature and torsion; osculating cone. The limiting position of the circle through three neighbouring points P, Q, R of a curve s, when Q and R coincide with P, clearly lies in the osculating plane at P. This circle, its centre C and its radius ρ are termed the **circle, centre** and **radius of curvature** of s at P.

If we make R first coincide with P, it follows that the circle of curvature is the limit of the circle which touches s at P and passes through Q, and therefore C lies on the normal plane at P. But, if we make R first coincide with Q, the circle of curvature is the limit of the circle which touches s at Q and passes through P. The centre of this last circle lies on the normal plane at Q. Hence the centre of curvature C, which lies on the normal plane at P, is the limit of a point on the normal plane at Q. It must therefore lie on the ultimate intersection of the normal planes at P and Q and so is the intersection of the osculating plane at P with the corresponding generator of the polar developable.

If, in like manner, μ, ν, π are three osculating planes of s, four right circular cones can be described, with $\mu\nu\pi$ as vertex, to touch them; if μ, ν are made to coincide with π, the ultimate intersection $\mu\nu\pi$ becomes P, where P is the point of s at which π is the osculating plane. Also, when μ, ν coincide with π, three of the above right circular cones shrink into line-cones along the tangent at P, but one remains a proper cone of finite vertical angle. This will be termed the **osculating cone** at P to either the curve s or its osculating developable; it is easily seen to touch the osculating plane along the tangent at P.

The same kind of argument which has been employed above to show that the centre of curvature lay on the ultimate intersection of two neighbouring normal planes which tend to coincide will show that the axis of the osculating cone is the ultimate intersection of neighbouring rectifying planes which tend to coincide. This axis is therefore what has been called the rectifying line in Art. 253. Thus the rectifying line at P passes through P.

If η is the angle between the binormals at P and Q, or, what is the same thing, between the osculating planes at P and Q, the limit of the ratio (arc $PQ)/\eta$ when Q tends to coincide with P is termed the **radius of torsion** of s at P, and is denoted by σ.

The sphere which has four-point contact with s at P is the **osculating sphere** at P, and its radius is the **radius of spherical curvature,** which we will denote by R. This sphere is the limit of the sphere through P and three other points Q, S, T of the curve s, when Q, S, T coincide with P. It is also the limit of the sphere which touches s at P and Q and therefore has its centre K on the generator of the polar developable corresponding to P. We may, however, go further than this. For by making T coincide first with P, or with Q, or with S, we find that K is the centre of a sphere which is the limit of all three spheres which pass through P, Q, S and touch s at one of these points. Thus the normal planes at each of P, Q, S pass through K in the limit, so that K is the ultimate intersection of three coincident tangent planes of the polar developable and so is the point corresponding to P on the cuspidal edge of this developable.

Since the centre of curvature C, as well as K, lies on the generator of the polar developable, which is perpendicular to the plane of the circle of curvature, a circle with C as centre and passing through P will lie entirely on the sphere centre K passing through P. Hence the circle of curvature is a small circle on the osculating sphere, whose spherical centre is the dyad determined by the diameter parallel to the binormal at P.

255. The spherical indicatrix: relations between radii of curvature and torsion. If through any point O half-rays OT, ON, OB are drawn parallel to the tangent, principal normal and binormal at a point P of s, in senses defined, in the case of the tangent by a prescribed sense of description of the curve, in that of the principal normal by the sense from P towards the centre of curvature, and in that of the binormal by a right-handed rotation of ON through a right angle about OT, these half-rays meet a given sphere of centre O and arbitrary radius at points T, N, B (Fig. 79). The successive positions of the trirectangular spherical triangle TNB on the sphere, as P moves along the curve s, constitute the **spherical indicatrix** of s.

Let now $T'N'B'$ be a near position of TNB, corresponding to a point P' on the curve near to P. The arcs TN, NB, BT represent, on the indicatrix, the osculating, normal and rectifying planes at P. If these arcs meet the corresponding arcs $T'N'$, $N'B'$, $B'T'$ at X, Y, Z respectively, then, since the ultimate intersections of the corresponding planes are the tangent at P, a line parallel to the binormal at P, and the axis of the osculating cone at P, the points X, Y, Z approach T, B, A in the limit, where A represents on the indicatrix the axis of the osculating cone. Also, the angles at X and Y being very small, the arcs TT', BB' (not shown in Fig. 79) diverge very little from the arcs $T'X$, XT and BY, YB', and so are approximately at right angles to both BT and $B'T'$. Thus, if ϕ is the angle at Z, by a well-known result in spherical geometry, we have, to a first approximation

$$\text{arc } TT' = \phi . \sin ZT$$

and

$$\text{arc } BB' = \phi . \sin ZB = \phi . \cos ZT.$$

By division

$$\frac{\text{arc } TT'}{\text{arc } BB'} = \tan ZT.$$

But if θ be the angle between the tangents at P, P' (which are here

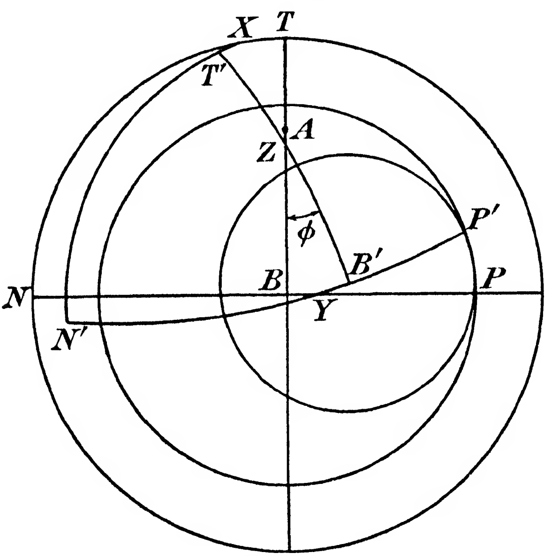


Fig. 79.

not the points shown in Fig. 79) and η the angle between the binormals

$$\frac{\text{arc } TT'}{\text{arc } BB'} = \frac{\theta}{\eta} = \left(\frac{\text{arc } PP'}{\eta}\right) \Big/ \left(\frac{\text{arc } PP'}{\theta}\right)$$

Thus

$$\left(\frac{\text{arc } PP'}{\eta}\right) \Big/ \left(\frac{\text{arc } PP'}{\theta}\right) = \tan ZT$$

and, proceeding to the limit

$$\sigma/\rho = \tan AT = \tan \alpha \text{ ....} (1)$$

where 2α is the vertical angle of the osculating cone. This is a well-known formula connecting σ and ρ.

Consider now a curve s which lies on a sphere; then this sphere is the osculating sphere at every point of the curve, and may be taken as indicating sphere; P, P' are now points on the sphere itself and they are the points shown in Fig. 79. Since the normal plane at P passes through the centre of the osculating sphere, P lies on the arc BN (Fig. 79) and similarly P' lies on the arc $B'N'$. Also B and B' are the spherical centres of the circles of curvature at P, P'. And since the circle of curvature at P is the limit of the small circle touching the curve at P and passing through P', $BP = BP'$ to the first order of approximation.

With the convention adopted, P, P' are on opposite sides of B, B' to N, N' respectively. Thus the difference between the angular radii BP, $B'P'$ of the circles of curvature is to the first order equal to BB'. The actual radii are given by

$$\rho = R \sin BP \quad \dots\dots\dots\dots (2)$$
$$\rho' = R \sin (BP - BB')$$

so that, to the same order of approximation,

$$\begin{aligned} \rho' - \rho &= -R \cos BP . \text{arc } BB' \\ &= -R\eta . \cos BP. \end{aligned}$$

Dividing by arc PP'

$$\frac{\rho' - \rho}{\text{arc } PP'} = -\frac{R\eta}{\text{arc } PP'} \cos BP$$

and in the limit

$$\frac{d\rho}{ds} = -\frac{R}{\sigma} \cos BP$$

or

$$\sigma \frac{d\rho}{ds} = -R \cos BP \quad \dots\dots\dots\dots (3)$$

Squaring and adding (2) and (3) we obtain

$$R^2 = \rho^2 + \sigma^2 \left(\frac{d\rho}{ds}\right)^2 \quad \dots\dots\dots\dots (4)$$

for any curve lying on a sphere.

If now two curves have three-point contact at a point P, they have the same osculating plane and circle of curvature at P. If they have four-point contact at P, they have also the same osculating sphere at P.

But further a curve k which has four-point contact with s at P is the limit of a curve k' passing through P and through three neighbouring points Q, S, T on s. Making the three latter points coincide at Q, the curve k' has the same osculating plane and circle of curvature at Q as the curve s. In the limit k has the same osculating plane and circle of curvature as s, to the first order approximation, at points in the immediate neighbourhood of P, as well as at P itself. Thus k has also (i) the same radius of torsion, (ii) the same $\frac{d\rho}{ds}$, as s.

By taking the curve k to lie on the osculating sphere of s at P, the formula (4) holds for it. But since every quantity occurring in the formula is the same for k and s, the formula also holds for s, and therefore for any curve whatever. Formula (4) then gives a second fundamental relation between R, ρ, σ.

It is sometimes stated * that there is no actual circle connected with the curve, whose radius is equal to the radius of torsion. This, however, is not the case; for formula (1) shows that if with P as centre a sphere is described to pass through the centre of curvature C and to meet the axis of the osculating cone at U, the tangent plane at U to the sphere just mentioned meets the osculating cone in a circle, whose radius is the radius of torsion.

256. Quadrics. A **quadric** is a surface of the second order. Every plane section of a quadric is a conic. There are three main types of quadrics, according to the nature of their intersections with the plane at infinity. The quadrics of the first type do not meet this plane in real points; they lie entirely at a finite distance and every plane section of them is an ellipse; they are called **ellipsoids.** The quadrics of the second type meet the plane at infinity in real conics but do not touch it, and are called **hyperboloids.** The quadrics of the third type, which touch the plane at infinity, are called **paraboloids.** Subclasses of these exist, which will be described more fully in Art. 257.

Notice that the sphere is a special case of the ellipsoid and the (real) cone of the second order a special case of the hyperboloid.

A quadric being a surface, we shall denote it by a Greek letter, *e.g.* ψ. The equation of a surface of second order contains *ten* coefficients, the *nine* ratios of which determine the equation. **A quadric is therefore, in general, determined by nine points.**

257. Generators and tangent planes of a quadric. Consider a point P on a quadric ψ. Let π be the tangent plane to ψ at

* *See* Salmon, "Geometry of Three Dimensions," 4th ed., p. 335.

P. Then π meets ψ in a conic s. But since every line through P in π is tangent to ψ, it is also tangent to s, which must accordingly reduce to a line-pair p, q. Thus through any point P of the quadric there pass two lines p, q which lie entirely in it. These lines p, q are termed **generators** of the quadric.

Let P_0 be a given point of ψ; p_0, q_0 the two generators through P_0. If P_1 be any other point on the generator p_0, then p_0 is one of the two generators through P_1. The second generator is a line q_1. Also q_1 cannot intersect q_0, for then p_0, q_0, q_1 would be the sides of a plane triangle and we should have a quadric intersecting a plane in a triangle, which is impossible.

Hence all the generators q_1, which intersect p_0, do not intersect one another.

Similarly all the generators p_1, which intersect q_0, do not intersect one another.

The two generators through any point Q of the quadric belong one to the system p and the other to the system q. This is clear from what has just been proved if Q lies on p_0 or q_0. If Q do not lie on p_0 or q_0, the plane p_0Q meets the quadric in a conic, which consists partly of p_0 and so must be a line pair. The other line of the pair, on which Q must lie, is coplanar with p_0, and therefore meets p_0 at a point R. Thus Q lies on the generator of the system q through R. Similarly Q lies on one generator of the system p.

It follows that each of the two sets of generators p, q contains all the points of the quadric.

Further, every generator p meets every generator q. For let p_1, q_1 be the two generators through any point P_1 of the quadric, and q_2 any other generator of the system q. As above, the plane P_1q_2 meets the quadric in a line-pair, of which q_2 is one member. The other is a generator which must meet q_2 and also pass through P_1 and therefore is one of the generators through P_1. Since q_1, q_2 cannot meet, it must be p_1. Thus q_2 must meet any generator p_1 of the system p.

If one generator, say p, of a real quadric is real, then the second generators q at the real points of p are necessarily real, and the generators p through the real points of a real generator q are likewise real. Examination of the arguments given previously shows that the generators p and q through any real point of the quadric are then real.

It follows that, if the two generators through any real point of the quadric are imaginary, the quadric can have no real generators. On the other hand a quadric with real generators has any number of imaginary generators, but these pass through imaginary points of the quadric.

If a quadric has real generators, the points at infinity on these generators are real points of the quadric. Such a quadric cannot then be an ellipsoid. On the other hand a quadric may have real points at infinity and not have real generators. If the quadric be a hyperboloid and the tangent planes through the points at infinity meet the quadric in real lines, the quadric has real generators and is called a **hyperboloid of one sheet**; but if they do not meet the quadric in real lines, the quadric is called a **hyperboloid of two sheets.** The reason for these names will be apparent later.

In the case of paraboloids, we have also two classes, according as the plane at infinity meets the quadric in real, or in imaginary, lines at infinity. In the first case we are said to have a **hyperbolic,** in the second case an **elliptic paraboloid.**

The imaginary generators of a sphere have, however, an important property, namely, that they are the circular lines through P in the tangent plane at P. For clearly they must pass through the points at infinity on the sphere, lying in the tangent plane at P, and these points must be on the circle at infinity (Art. 243) and therefore circular points.

An important particular case is when two generators through a point P of the quadric are coincident. In this case the quadric must reduce to a cone of the second order. This we can prove as follows.

Let (p, p) be the coincident generators through P, Q any other point of the quadric ψ, not lying on p. As before, the plane pQ meets the quadric in a line pair (p, q) and q both meets p and passes through Q. Let it meet p at V. Let r be the second generator through Q, and let it meet the tangent plane π at P in R. Since R is a point of the quadric ψ lying in π and all the points of ψ in π lie on p (doubled), R is a point of p, so that r meets p at R. Thus, if R is distinct from V, we have a triangle VQR lying entirely in a quadric, which is impossible, unless ψ breaks up into two planes. Hence r must pass through V.

Thus all the generators of the quadric are double and any two of them intersect. Therefore, by Art. 7, they either all pass through the same point V, in which case the quadric reduces to a cone of the second order having V for vertex, or they all lie in one plane in which case the quadric reduces to a pair of coincident planes.

In the alternative, when ψ breaks up into two planes, we note that a pair of planes is a particular case of a cone of the second order; in this case there is a whole pencil of generators through any point of the quadric, and two such pencils through any point common to the two planes. If any generator other than this line common to the two planes counts twice, the plane-pair must reduce to a pair of coincident planes.

258. Focal spheres. A well-known property of the foci of a conic can be at once deduced from the result that the generators of a sphere are circular lines.

Let κ be a right circular cone, σ a sphere touching κ along a circle c, α any plane touching σ at F and meeting κ in a conic s. Then F is a focus of s. If x, y are the generators of σ through F, these generators clearly touch κ at the points I, J where they meet the circle c, which they must meet, since x, y, c lie in the sphere σ. Hence, since they lie in α, they must touch the intersection of α and κ, that is the conic s. Hence x, y are the two tangents from F to s.

But x, y, being generators of a sphere, are circular lines in α. Hence, by Art. 145, F is a focus of s. Moreover I, J are clearly points common to α, κ and σ, that is, to α and c. Accordingly they must lie on the intersection of α with the plane of the circle c. But I, J are the points of contact of x, y with κ, and therefore with s, so that IJ is the polar of F with respect to s, and so is the directrix corresponding to F.

We have therefore the following construction for the foci of a plane section of a right circular cone: describe the spheres touching the cone and the plane; their points of contact are foci.

Two such spheres are real, namely those which touch the cone internally; their points of contact are the real foci. If the plane meets one half-cone only, the spheres lie on the same side of the vertex of the cone, but on opposite sides of the plane; the conic s is then an ellipse. If the plane meets both half-cones, the spheres lie on opposite sides of the vertex, but on the same side of the plane; s is then a hyperbola. If the plane is parallel to a generator, only one proper real sphere exists and s is a parabola.

259. Reguli. The generators p of Article 257 are said to form a **regulus** on the quadric ψ. Similarly the generators q form a regulus.

The two sets are said to be **complementary reguli.** The lines of either regulus may be spoken of as **transversals** of the other.

A regulus determines on any two transversals homographic ranges.

For let a, b, c, d be any four lines of the regulus, x, y two transversals belonging to the complementary regulus. Let a, b, c, d meet x at A_1, B_1, C_1, D_1 and y at A_2, B_2, C_2, D_2. Cut the quadric ψ to which the reguli belong by a plane α. This meets ψ in a conic s, which meets a, b, c, d, x, y at A, B, C, D, X, Y respectively.

The planes $x\ (abcd)$ form an axial pencil which meets α in the flat pencil $X(ABCD)$. Similarly $y(abcd)$ form an axial pencil meeting α in $Y(ABCD)$.

Because X, Y, A, B, C, D lie on a conic

$$X\{ABCD\} = Y\{ABCD\},$$

and therefore

$$x\{abcd\} = y\{abcd\}.$$

Cutting the axial pencil of axis x by y, and that of axis y by x, we have at once

$$\{A_2B_2C_2D_2\} = \{A_1B_1C_1D_1\},$$

which shows that the ranges on the two transversals are equianharmonic and therefore homographic.

This common cross-ratio may be called the **cross-ratio of the four lines of the regulus.** The regulus, like the range, the flat pencil and the axial pencil, is one of the standard forms. It may be reckoned as a form of the second order, since it lies in a surface of that order.

Two reguli will be said to be homographic if corresponding lines can be related by a one-one algebraic correspondence. They meet any two planes in homographic ranges of the second order and any generators of their complementary reguli in homographic ranges of the first order.

Not only does a regulus determine homographic ranges on two transversals x, y, but it determines with x, y two axial pencils homographic with these ranges and with one another. This follows immediately from the proof given above, since we have seen that

$$x\{abcd\} = y\{abcd\} = \{A_1B_1C_1D_1\} = \{A_2B_2C_2D_2\}.$$

Reguli will be said to be cobasal if they belong to the *same* set of generators of a quadric. Two homographic cobasal reguli have two self-corresponding lines, which may be real, coincident or imaginary.

Similarly we may have an involution regulus, which has two distinct double rays, real or imaginary.

260. Quadric as product of homographic ranges or axial pencils. If, in the theorem of the last Article, we denote by p a variable line of the regulus which meets x and y at P_1, P_2 we have proved that

$$[P_1]\barwedge[P_2]$$

and $$x[p]\barwedge y[p].$$

Thus any generator p of a quadric (other than a cone) is (i) the join of corresponding points of two homographic ranges on two generators of the complementary system, (ii) the meet of corresponding planes of two homographic axial pencils through two generators of the complementary system.

The quadric is therefore obtained as the product of two homographic ranges on skew lines, or of two homographic axial pencils with non-intersecting axes.

Conversely, any such product must necessarily be a quadric.

Take the second case first. Let π, π' be corresponding planes of the axial pencils, meeting any straight line l at P, P'. Then $[P]\barwedge[\pi]\barwedge[\pi']\barwedge[P']$. The ranges $[P]$, $[P']$ have two self-corresponding points, which are the intersections of l with the locus: the latter is therefore of the second order, and thus a quadric.

The first case is immediately reducible to this; for, let $P_1P_2=p$ and let the bases of $[P_1]$, $[P_2]$ be x, y respectively. Then p is the intersection of corresponding planes in the homographic axial pencils $x[P_2]$, $y[P_1]$ and generates a quadric by the preceding.

We may notice that the quadric, in both cases, contains the bases x, y, and that the points P_1, P_2 are the points of contact of the planes xp, yp, which are tangent planes to the quadric. Hence we obtain the following theorem: **the tangent planes to a quadric at the points of a generator form an axial pencil homographic with the range of their points of contact.**

We are now in a position to free our definition of the regulus from any necessary connection with a quadric. For let x, y, z be three non-intersecting lines in space (called **directrices**). Take any point P of x. The plane Pz meets y at one point Q only and PQ meets z at a unique point R. There is accordingly a unique straight line PQR meeting x, y, and z.

That this determines a regulus according to the previous definition is immediately obvious. For the relation between P and Q is clearly one-one and algebraic; therefore $[P] \barwedge [Q]$ and PQ is a generator of a quadric and therefore a line of a regulus.

Incidentally we see that **a quadric is uniquely determined by any three generators of the same system.**

EXAMPLES

1. Show that a regulus projects from any point upon any plane into a homographic pencil of the second order.

2. Two fixed straight lines a and b meet a conic s, but are not coplanar with s or with each other. Show that a straight line which meets s, a, b describes a quadric.

3. Two skew lines a, b meet a conic k at points A, B, but are not in the plane of the conic. If the unique transversal to a and b from a point P of the conic meets them at Q, R respectively, prove that, when P describes the conic, Q and R describe projective ranges.

In the case when k is a parabola, and a and b are parallel to a plane γ through the axis of the parabola, prove that the ranges $[Q]$, $[R]$ are similar; and that any plane parallel to γ is met by the transversals PQR in the points of a straight line.

4. If k_1, k_2 are two plane sections of a quadric ψ, and any generator of ψ, of one system, meets k_1 at P_1 and k_2 at P_2, prove that $[P_1]^2 \barwedge [P_2]^2$.

261. Homographic complementary reguli. Consider a plane section s of a quadric ψ. If we relate corresponding rays p, p' of two complementary reguli on ψ, so that they intersect at a point P of s, then these reguli will be homographic.

For a generator p of ψ cannot meet s at more than one point, otherwise the plane of s would meet ψ in both s and p, which is impossible. Hence, p being known, P, and therefore p', is uniquely determined; and conversely. Therefore the reguli are homographic.

Conversely the product of two homographic complementary reguli is a conic section of the quadric ψ in which the reguli lie.

For let a, b, c and a', b', c' be the three (arbitrarily selected) corresponding rays which define the homography. Let $A = aa'$, $B = bb'$, $C = cc'$, then BC, CA, AB are not generators of ψ, and the plane ABC meets ψ in a proper conic s.

If now p, p' be two corresponding rays of the reguli, meeting s at P, P', we have $a'[p] \barwedge a[p']$, so that, taking sections of these axial pencils by the plane ABC, $A[P] \barwedge A[P']$ and $[P]^2 \barwedge [P']^2$.

But the last two ranges of the second order have clearly A, B, C for three self-corresponding points. Hence the ranges must coincide, and $P = P'$ or the corresponding rays p, p' meet on the conic s.

262. Class of a quadric. Let a quadric be defined by the homographic ranges $[P]$, $[P']$ determined by one of its reguli on two generators x, x' of the other. Let u be any straight line. The cobasal homographic axial pencils $u[P]$, $u[P']$ have two self-corresponding planes. Each one of these contains a generator PP' of the quadric. It therefore contains a second generator and touches the quadric at their intersection.

Thus through any straight line u two tangent planes can be drawn to a quadric or a quadric is a surface of the second class.

Conversely every surface of the second class is a quadric. For, by the principle of duality in space, the reciprocal of any surface of the second class is a surface of the second order. Since there are two tangent planes to this latter surface through an arbitrary line, any line u will meet the original surface of the second class at two points. Hence this surface is a quadric.

263. Degenerate quadrics. Precisely as the conic, considered as a locus, may degenerate into a line-pair, or, considered as an envelope, may degenerate into a point-pair, so the quadric may degenerate in different ways, according as we consider it generated by two homographic axial pencils, or by two homographic ranges.

Taking two axial pencils, the first type of degeneration which presents itself is when the axes intersect, that is, the bases of the axial pencils are coplanar. Corresponding planes will then (Art. 240) meet in generators of a cone of the second order, which is thus one type of degenerate quadric. It retains the typical property of the quadric that it is met (in general) by any straight line in two points ; but it is no longer true that two tangent planes can be drawn to it through any given line. This is only possible when the line passes through the vertex. In all other cases the plane through the given line and the vertex of the cone has to be regarded as a double tangent plane, in order to maintain artificially the quadric property.

If, further, the two axial pencils have a self-corresponding plane, the other corresponding planes meet on a fixed plane, and the cone of the second order breaks up into a **plane-pair.** This is still a locus of the second order. In this case no tangent plane can in general be drawn through an arbitrary line, even with the very special interpretation given in the last paragraph. No *class* can therefore be ascribed to the plane-pair.

Now consider two homographic ranges. If their bases are made coplanar, the generators of the quadric lie in a plane, which they entirely fill up, with the exception of the inside of this conic envelope. We may think of the quadric surface as pressed flat, so as to form a double-sheeted plane, with a hole in it in the shape of the conic, the two sheets joining up on the edge of the hole. Such a quadric will be called a **disc quadric.** A conic, so considered, is therefore a degenerate quadric. It remains a surface of the second class, for two tangent planes can be drawn to it, in general, through any line in space, but it can only be considered of the second order if, by an artificial convention, we consider an intersection with the plane of the conic to be double, in view of the two sheets above mentioned.

If the two coplanar ranges have a self-corresponding point, the conic in question itself degenerates into a point-pair, the hole shrinking to a slit in the plane, or the disc quadric to a thin rod, connecting the points of the pair. This does, indeed, still give us an envelope of the second class.

It is also clear that the cone of the second order can degenerate into a line-pair, and so can the conic. But for the cone of the second order to degenerate into a line-pair, it must first arise as the product of two homographic flat pencils of a star; the product of two homographic axial pencils of a star cannot produce a line-pair. On the other hand, a conic obtained as the product of homographic ranges cannot degenerate into a line-pair. If therefore we start from the quadric as above, we cannot arrive at the line-pair.

Moreover it is clear that, in general, a straight line in space does not meet a line-pair at two points, nor can two planes be drawn through it to touch the line-pair. The line-pair, considered as a three-dimensional locus, is neither of the second order, nor of the second class (although it is of the second *degree*), and has no claim to be considered as even a degenerate quadric.

And this indeed is borne out by analytical considerations, for whereas the cone and plane-pair can be represented by a single equation in point-co-ordinates, and the conic and point-pair by a single equation in plane-co-ordinates, the line-pair cannot be so represented, but always needs *two* equations to specify it.

264. Pole and polar plane. Let P be any point and let any ray through P meet a quadric at R, S. If P' be the point harmonically conjugate to P with regard to R, S, then P' lies on a fixed plane.

For take two rays through P, PR_1S_1, PR_2S_2 and let $P_1'P_2'$ be the corresponding positions of P'. Join $P_1'P_2'$. Then if α be the plane of the two rays PR_1S_1, PR_2S_2, $P_1'P_2'$ is the polar of P with regard to the conic in which α meets the quadric. Hence the locus of points P' corresponding to all rays through P which lie in α is the straight line $P_1'P_2'$. Thus the straight line joining any two points on the locus lies entirely in the locus. But this property defines a plane.

This plane is called the **polar plane** of P with regard to the quadric.

When the points R, S coincide, P' coincides with them. Thus **the tangent cone from P touches the quadric along a plane section.** This cone is therefore of the second order.

If P lie on the polar plane ρ of R, and PR meet the quadric at (S, T), then (P, R) are harmonic conjugates with regard to S, T and therefore the polar plane π of P passes through R.

P, R are **conjugate points** and π, ρ **conjugate planes** with regard to the quadric.

Consider the poles of planes through P. These lie on its polar plane π. Similarly the poles of planes through R lie on ρ. Thus the poles of planes through PR lie on a fixed line $\pi\rho$.

Hence if S be any point of PR, S' any point of $\pi\rho$, SS' is harmonically divided by the quadric.

The symmetry of this last relation shows that the poles of planes through $\pi\rho$ lie on PR.

Two such lines PR, $\pi\rho$ are said to be **polar lines** with regard to the quadric.

The polar plane of a point P on the quadric is the tangent plane at P. For the polar plane of every point R in the tangent plane at P passes through P.

Conjugate lines with respect to a quadric are lines such that each meets the polar line of the other. For, if p meets the polar line q' of q, pq' determine a plane π whose pole P lies on q. But, since p lies on π, the polar line p' of p passes through P, therefore q meets p', and the condition is symmetrical, as stated. Clearly, if P is any point, π its polar plane with respect to the quadric, any line through P is conjugate to any line of π.

A line is also said to be conjugate to any point in its polar line, and to any plane through its polar line.

If x, y are polar lines, then y is the chord of contact of tangent planes through x. For let σ, τ be the tangent planes through x, touching the quadric at S, T. Since σ is the polar plane of S and τ the polar plane of T, ST is the polar line of $\sigma\tau$, that is, of x.

If π, ρ are two other planes through x, conjugate for the quadric, their poles P, R lie on the polar line ST; but P lies on ρ and R lies on π, therefore P, R are the meets of ST with ρ, π respectively. Because P is the pole of π, ST is harmonically divided by P and π, that is, σ, τ are harmonically conjugate with respect to π, ρ. That is, **two conjugate planes are harmonically conjugate with respect to the two tangent planes through their intersection.**

It follows that conjugate planes through a line x form an involution axial pencil, of which the double planes are the tangent planes to the quadric through x. Similarly conjugate points on a line x form an involution of which the double points are the intersections of the quadric with x. As in Chapter IV, the polar planes π of points P of a range on a straight line form an axial pencil homographic with the range, the bases of the two forms being polar lines.

If we cut the axial pencil $[\pi]$ by a plane α, and join the pole A of α to the range P, we obtain two homographic flat pencils $\alpha[\pi]$, $A[P]$, in which corresponding lines are polar lines. Thus, **to any flat pencil corresponds homographically the flat pencil of its polar lines.** The planes of two such pencils are conjugate planes, since A, the pole of the plane α, lies in the plane of the other pencil, and likewise the vertices are conjugate points.

Two such polar flat pencils cannot be cobasal unless their plane α is a tangent plane and their vertex A its point of contact. In this case a pair of polar lines are mates in an involution, of which the generators through A are the double lines.

EXAMPLES

1. If two polar lines intersect, prove that their plane touches the quadric, with their intersection as the point of contact.

2. If a skew quadrilateral is formed of four generators of a quadric, two of each system, prove that the joins of opposite vertices are polar lines for the quadric.

3. Show that polar lines for a sphere are perpendicular.

4. Prove that a line which is its own polar line with respect to a quadric is a generator of the quadric, and, conversely, that every generator of the quadric is its own polar line.

5. Show that two planes conjugate for a quadric are conjugate for every tangent cone whose vertex lies on their intersection.

6. If two intersecting lines are conjugate for a quadric, prove that they are conjugate for (1) the conic intersection of the quadric by the plane through the lines, (2) the tangent cone to the quadric from the intersection of the lines.

7. If a line is self-conjugate for a quadric, prove that it is a tangent line to the quadric.

265. Twisted cubic. A **twisted cubic** is a curve of the third degree: it may be obtained as the product of three homographic axial pencils. For take any three chords a, b, c of the twisted cubic. Let P be any point on the curve; denote the planes aP, bP, cP by π_1, π_2, π_3. Now since a already meets the cubic at two points (being taken a chord), a plane π_1 through it can meet the cubic again at one point P only. Thus when π_1 is given, P, and therefore π_2 and π_3, are uniquely determined. Similarly if π_2 or π_3 be given, the other two are uniquely determined. Hence $[\pi_1]$, $[\pi_2]$, $[\pi_3]$ are three homographic axial pencils of planes, of which the twisted cubic is the product.

Conversely **any three homographic axial pencils** $[\pi_1]$, $[\pi_2]$, $[\pi_3]$, whose axes are a, b, c, **determine in general a twisted cubic as their product.** For they determine on any plane λ three homographic flat pencils $[p_1]$, $[p_2]$, $[p_3]$ having for vertices the points A, B, C in which a, b, c meet λ. $[p_1]$, $[p_3]$ have as their product a conic s_1 passing through A and C; $[p_2]$, $[p_3]$ have as their product a conic s_2 passing through B and C. If P is a point of intersection of s_1 and s_2, other than C, P lies on three mutually corresponding rays of $[p_1]$, $[p]_2$, $[p_3]$ and therefore on three mutually corresponding planes of $[\pi_1]$, $[\pi_2]$, $[\pi_3]$. It is therefore a point of the product-locus. Since s_1, s_2 have three intersections other than C, there are three such points P.

C is not, in general, a relevant point, unless AC, BC happen to correspond to the same line p_3 through C, in which case s_1 and s_2 touch along this line. But s_1, s_2 can then have only two other intersections so that, in every case, there can be only three points of λ lying on the product-locus, which is accordingly a twisted cubic.

If a, c intersect and $[\pi_1]$, $[\pi_3]$ have a self-corresponding plane β, their product degenerates (*see* Art. 240) into β and another plane δ. If β_2 is the plane of $[\pi_2]$ corresponding to β, every point of $\beta\beta_2$ is a point common to three corresponding planes of $[\pi_1]$, $[\pi_2]$, $[\pi_3]$. Further, the product of $[\pi_1]$ $[\pi_2]$ is a quadric ψ which meets δ in a conic k, every point of which lies on the product-locus of the three axial pencils. The twisted cubic then degenerates into the straight line $\beta\beta_2$ and the conic k, which meets $\beta\beta_2$ at the point ac.

If, further, a and b intersect and $[\pi_1]$ and $[\pi_2]$ have a self-corresponding plane γ, the product $[\pi_1\pi_2]$ breaks up into γ and another plane ϵ. If γ_3 is the plane of $[\pi_3]$ corresponding to γ, then the three straight lines $\beta\beta_2$, $\gamma\gamma_3$, $\delta\epsilon$ form the locus; $\beta\epsilon$, $\gamma\delta$ are not relevant, unless $\beta\epsilon$ lies in β_2 or $\gamma\delta$ lies in γ_3, in which case they coincide with the lines previously found. The twisted cubic then degenerates into three straight lines.

A non-degenerate twisted cubic cannot meet a straight line at more than two points. For if it meet a straight line a at points A, B, C, take a point D of the cubic outside the line. Then the cubic would meet the plane aD at four points A, B, C, D which is impossible.

It is clear that **every twisted cubic lies on a quadric,** for the twisted cubic just considered lies on the product of $[\pi_1]$ $[\pi_2]$, which is a quadric of which a, b are generators. Similarly it lies in the product of $[\pi_1]$ $[\pi_3]$, of which a, c are generators.

Reciprocating the above properties, we see that **the plane through a set of corresponding points of three homographic ranges which do not all lie in one plane touches a developable of the third class,** which may in special cases degenerate into a straight line and a cone of the second order, or into three straight lines.

Examples

1. Show that a twisted cubic is entirely determined by any six points on it.

[For if A, B, C, P, Q, R be the six points, the homographic relation between the axial pencils passing through BC, CA, AB respectively is entirely determined by the triads containing P, Q, R.]

2. Prove that any four given points of a twisted cubic determine with any variable chord of the cubic an axial pencil of constant cross-ratio.

3. If A, B, C, P, Q, R be six points on a twisted cubic, show that the tangent at A to the cubic lies in the planes through AB, AC which correspond to the plane ABC of the axial pencil through BC in the correspondence between homographic axial pencils determined by the triads of planes joining BC, CA, AB to the points P, Q, R.

4. If A is a point on a twisted cubic, a the tangent at A, P, Q, R, S four other points of the cubic, prove that the osculating plane to the cubic at A is the tangent plane along a to the cone of the second order determined by the five generators AP, AQ, AR, AS and a.

5. Prove that the chords of a twisted cubic through a given point O of the curve lie on a cone of the second order.

6. Prove that a twisted cubic lying in a quadric meets the generators of one set in two points, and the generators of the other set in one point only.

[Consider the intersections of the cubic by a tangent plane to the quadric: two must lie on one generator and one on the other.]

7. Prove that two given points of a twisted cubic lying in a quadric determine, with the generators of each system, homographic axial pencils.

8. Prove that a regulus and an axial pencil homographic with the regulus generate a twisted cubic.

266. Intersections of quadrics. If ψ_1, ψ_2 be two quadrics, of which q is the intersection, then any plane π meets ψ_1, ψ_2 in two conics s_1, s_2, and q in the intersections of s_1, s_2. Since the latter are four in number, every plane meets q in four points so that q is, in general, a **twisted quartic.**

If ψ_1, ψ_2 have one generator x in common, this generator is part of the locus q. The remainder of q meets any plane in three points and so must be a twisted cubic. This can also be seen otherwise, for if x_1, x_2 be two other generators of ψ_1, ψ_2 respectively, belonging to the same system as x in each case, ψ_1 is obtained as the product of homographic axial pencils $[\pi_1]$, $[\pi]$ through x_1, x respectively, and ψ_2 as the product of homographic pencils $[\pi]$, $[\pi_2]$ through x, x_2 respectively. The points common to ψ_1, ψ_2 are therefore the intersections of corresponding planes $[\pi]$, $[\pi_1]$, $[\pi_2]$, that is they lie on a twisted cubic of which x, x_1, x_2 are chords.

If ψ_1, ψ_2 have two generators x, x' of the same system in common, the planes π_1, π_2 of the last paragraph may be taken to pass through x'; π, π_1, π_2 will not in general intersect outside x', unless π_1, π_2 coincide. This will happen when π_1, π_2 coincide with either of the self-corresponding planes α, β of the cobasal axial pencils $[\pi_1]$, $[\pi_2]$. If σ, τ are the planes of π corresponding to α, β respectively, then every point of $\sigma\alpha$ or $\tau\beta$ is on three corresponding planes : these two lines, together with x, x', give the whole intersection. Also $\sigma\alpha$, $\sigma\beta$ meet both x and x' and so are generators y, y' of the other system. The intersection of ψ_1, ψ_2 then consists of a quadrilateral consisting of two generators of each system.

If ψ_1, ψ_2 have part of their intersection in a plane, then this part must be a conic s, which may degenerate into a pair of generators x, y of opposite systems, and this is part of the locus q. Since a conic meets any plane at two points, the remainder of the locus q is a curve of the second degree, that is, another conic k, or two lines, which cannot be generators of the same system ; the twisted quartic then breaks up into two conics which may, or may not, degenerate into line-pairs.

Note that, since the conic k is a plane curve, if two quadrics have one common plane section, they have a second common plane section.

When the intersection consists of a quadrilateral formed by two generators x, x' of one system and two generators y, y' of the other, these lines may be associated in coplanar pairs either as x, y ; x', y' or as x, y' ; x', y, so that there are now four common plane sections.

In the more general case, if α, β are the planes of the sections s, k, then s, k must intersect on $\alpha\beta$. If then C, D are the points where $\alpha\beta$ meets both quadrics, the tangents to s, k at C are tangent lines to both quadrics and the plane through them is a common tangent plane at C to both quadrics. Similarly the quadrics have a common tangent plane at D.

Conversely, if two quadrics ψ_1, ψ_2 touch at two points C, D and P is any other point on their intersection, not lying in CD, the plane PCD meets the quadrics in two conics k_1, k_2, both of which pass through C, D and P. If now the common tangent planes at C and D do not contain CD, k_1, k_2 touch at C, D and have the point P common. Hence they coincide entirely, so that the intersection of ψ_1, ψ_2 contains one conic, and therefore two conics.

If, however, the common tangent plane at C contains CD, then CD is a tangent line to both quadrics, and, since it passes through another point D common to ψ_1, ψ_2, it is a common generator of ψ_1, ψ_2. The conics k_1, k_2 then degenerate into line-pairs, CD being a component of each pair, the other components being the second generators in the plane PCD; these intersect at P. In this case the common tangent plane at D also contains CD. But k_1, k_2 do not coincide as a whole and the previous conclusion ceases to hold.

If the two quadrics touch at three points A, B and C, and none of BC, CA, AB is a generator, the plane ABC cuts them in conics which touch at each of A, B C and therefore coincide. The meet of the tangent planes at A, B, C is then the pole V of the plane ABC with regard to both quadrics, and the cone formed by joining V to the points of the conic in which the plane ABC meets both quadrics is the tangent cone from V to both quadrics. The latter therefore touch along the whole of the common conic.

By Art. 251 three quadrics ψ_1, ψ_2, ψ_3 will intersect in general in eight points. If the quadrics ψ_2, ψ_3 have a common generator x, the remainder of their intersection is a twisted cubic t. Since x meets ψ_1 at two points, then six of the eight intersections of ψ_1, ψ_2, ψ_3 belong to t. But every twisted cubic can be so obtained. Hence **every twisted cubic meets a quadric in six points.**

If ψ_1, ψ_3 have a common generator x_1 and ψ_2, ψ_3 have a common generator x_2 of the same system as x_1, the intersection of ψ_1, ψ_3 consists of x_1 and a twisted cubic t_1 lying in ψ_3, and the intersection of ψ_2, ψ_3 consists of x_2 and a twisted cubic t_2 lying in ψ_3. The eight points of intersection of the three quadrics then consist of: the two intersections of x_1 with ψ_2, the two intersections of x_2

with ψ_1, and four other points, which are the intersections of t_1 and t_2. Thus, **two twisted cubics lying on the same quadric and having the same system of generators as chords intersect, in general, in four points.**

Example

Show that if a twisted cubic and a quadric have seven points common they are altogether incident.

267. Homographic spaces. In precisely the same way as in a plane, so in space of three dimensions we can have a one-one algebraic correspondence between points, such that planes correspond to planes and therefore lines to lines. Such a relation may be described as a **space homography.** Two space fields Φ, Φ' thus connected are homographic, and, as in Art. 161, it may be shown that the relations between the point-co-ordinates are expressed by three equations of the form

$$x' = P_1/P_4,\ y' = P_2/P_4,\ z' = P_3/P_4,$$

where $\qquad P_r \equiv a_r x + b_r y + c_r z + d_r,\ (r = 1, 2, 3, 4).$

Similarly the equations connecting the plane-co-ordinates are

$$l' = Q_1/Q_4,\ m' = Q_2/Q_4,\ n' = Q_3/Q_4,$$

where $Q_r \equiv A_r l + B_r m + C_r n + D_r$, $(r = 1, 2, 3, 4)$, and A_r, B_r, etc. are the co-factors of a_r, b_r, etc. in the determinant of the coefficients a, b, c, d. These relations may also be expressed in the form

$$l = (a_1 l' + a_2 m' + a_3 n' + a_4)/(d_1 l' + d_2 m' + d_3 n' + d_4),$$

with two similar equations, and

$$x = (A_1 x' + A_2 y' + A_3 z' + A_4)/(D_1 x' + D_2 y' + D_3 z' + D_4),$$

with two similar equations.

In such a space homography corresponding fields in corresponding planes, as also all corresponding forms such as ranges, flat or axial pencils, ranges and pencils of the second order, reguli, etc. are homographic.

The planes $P_4 = 0$ and $D_1 x' + D_2 y' + D_3 z' + D_4 = 0$ correspond to the plane at infinity and are termed the **vanishing planes.** A space homography is entirely determined by five points corresponding to five given points, no four of each five lying in a plane, and therefore no three lying in a straight line.

To prove this, let A_1, B_1, C_1, D_1, E_1 correspond to A_2, B_2, C_2, D_2, E_2, in the homographic spaces Φ_1, Φ_2, no four points of either set being coplanar. Let P_1 be any point of Φ_1 not lying in the plane $A_1B_1C_1$, and let P_2 be the corresponding point of Φ_2.

Then, in the corresponding axial pencils of axes B_1C_1, B_2C_2

$$B_1C_1(A_1D_1E_1P_1)\barwedge B_2C_2(A_2D_2E_2P_2)$$

which determines uniquely the plane $B_2C_2P_2$ when $B_1C_1P_1$ is known, since no two of the three planes $B_1C_1A_1$, $B_1C_1D_1$, $B_1C_1E_1$ and no two of the three planes $B_2C_2A_2$, $B_2C_2D_2$, $B_2C_2E_2$ are coincident.

In like manner $C_2A_2P_2$, $A_2B_2P_2$ are determined. If P_1 is not in $A_1B_1C_1$, then P_2 is not in $A_2B_2C_2$ and the three planes above must be distinct, and cannot have a line in common since they pass through the sides of a plane triangle.

This fixes uniquely the corresponding points of all points outside the plane $A_1B_1C_1$.

That this construction leads to a homography is easily verified. For, if P_1 describe a line x_1, then the axial pencils $A_1B_1[P_1]$, $A_1C_1[P_1]$ are homographic and have the plane $A_1B_1C_1$ self-corresponding. Therefore, by the construction for P_2, the axial pencils $A_2B_2[P_2]$, $A_2C_2[P_2]$ are homographic and have the plane $A_2B_2C_2$ self-corresponding. Hence $A_2B_2P_2$, $A_2C_2P_2$ meet on a fixed plane λ. Similarly $A_2C_2P_2$, $B_2C_2P_2$ meet on a fixed plane μ. Thus P_2 describes the straight line $\lambda\mu$, which is thus x_2. Straight lines therefore correspond to straight lines, and this necessitates that planes correspond to planes, for if two lines P_1Q_1, R_1S_1 intersect, the corresponding lines P_2Q_2, R_2S_2 also intersect, so that four coplanar points correspond to four coplanar points. The space fields Φ_1, Φ_2 thus obtained are therefore homographic.

The above has left the points in the planes $A_1B_1C_1$, $A_2B_2C_2$ unrelated. It is clear that these two planes must correspond in the homography, since no point outside $A_2B_2C_2$ can correspond to a point on $A_1B_1C_1$. We can complete the correspondence by making intersections of corresponding lines with these planes correspond. If D_1E_1 meet the plane $A_1B_1C_1$ at F_1 and D_2E_2 meet the plane $A_2B_2C_2$ at F_2, then A_1, B_1, C_1, F_1 and A_2, B_2, C_2, F_2 form corresponding plane tetrads which define uniquely the correspondence between the fields in those planes (Art. 162).

Thus the two original sets of five points determine the homography completely.

268. Self-corresponding points of a space homography. It is obvious from the last Article that, in general, two homographic spaces Φ_1, Φ_2 cannot have more than four non-coplanar self-corresponding points, since five self-corresponding points, no four of which are coplanar, would determine the homography as an identity. We will now show that there are, in general, four such points.

Let a_1 be any straight line in Φ_1, not passing through a self-corresponding point, and let a_2 be its corresponding line in Φ_2. If $a_2 \equiv b_1$ in Φ_1 corresponds to b_2 in Φ_2, and $b_2 \equiv c_1$ in Φ_1 corresponds to c_2 in Φ_2, then $a_2 \equiv b_1$, $b_2 \equiv c_1$ and c_2 can contain no self-corresponding point. The lines a_1, a_2 might, as a special case, intersect, in which case b_1, b_2 and also c_1, c_2 will intersect, but these intersections must all be different, otherwise they must all coincide at a self-corresponding point, a case which has been excluded.

Now let α_1 be any plane through a_1, $\alpha_2 \equiv \beta_1$ the corresponding plane through $a_2 \equiv b_1$, $\beta_2 \equiv \gamma_1$ the corresponding plane through $b_2 \equiv c_1$, γ_2 the corresponding plane through c_2.

Then $\alpha_1\alpha_2$, $\beta_1\beta_2$, $\gamma_1\gamma_2$ are generators of three quadrics ψ_1, ψ_2, ψ_3 (which may as a special case be cones with distinct vertices). ψ_1, ψ_2 have $a_2 \equiv b_1$ as a common generator; ψ_2, ψ_3 have $b_2 \equiv c_1$ as a common generator; if ψ_1, ψ_2, ψ_3 are proper quadrics, b_1, b_2 are generators of ψ_2 of the same system.

We have therefore the case considered in the last paragraph of Art. 266 (which applies equally to proper quadrics and to cones, provided the cones have not common vertices). Hence ψ_1, ψ_2, ψ_3 have, outside b_1, b_2, four intersections A, B, C, D, which are not in general coplanar since they lie on a twisted cubic.

Every intersection A, B, C, D is common to six planes α_1, α_2, β_1, β_2, γ_1, γ_2, that is, corresponding points $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$ coincide at such a point. A, B, C, D are therefore self-corresponding points of the homography.

Clearly b_1 and b_2 cannot contain any self-corresponding points. Conversely, every self-corresponding point of the homography must lie in ψ_1, ψ_2 and ψ_3. For if P be such a point, the planes a_1P, a_2P correspond and so are planes α_1, α_2. Similarly b_1P corresponds to b_2P, so that if $b_1P = \beta_1$, $b_2P = \beta_2$; note that β_1 is then α_2. Again $c_1P = \beta_2 = \gamma_1$ and corresponds to $c_2P = \gamma_2$. P therefore lies on each of ψ_1, ψ_2, ψ_3 and so must be one of their points of intersection.

Two homographic space fields have therefore, in general, four non-coplanar self-corresponding points and four only. The faces of the tetrahedron formed by these four points are the self-corresponding planes and its edges are the self-corresponding lines.

The homography is determined if, in addition to the four self-corresponding points A, B, C, D, we are given a pair of corresponding points E_1, E_2, which do not lie on a face of the self-corresponding tetrahedron. But, if they do lie on such a face, say ABC, the space homography is no longer determined, but only the homography between the plane fields in ABC. Another pair F_1, F_2 are then required, but their positions can no longer be arbitrarily selected. For if DF_1 meet the plane ABC at G_1, G_2 is a known point, and F_2 must be taken on DG_2.

If, in such a case, the points E_1, E_2 are made to coincide at a point E, in the plane ABC but not on any side of the triangle ABC, the homography in the plane $ABC \equiv \pi$ has four self-corresponding points, no three being in line. In this case every point of π is self-corresponding; every pair of corresponding lines meet on π and every pair of corresponding planes meet in a line of π. Further, every line through D is self-corresponding, so that if P_1, P_2 are any two corresponding points DP_1, DP_2 coincide, and the joins of corresponding points pass through a fixed point D.

We have then an analogue of plane perspective, and it is reasonable to give it the name of **space perspective,** since any two corresponding plane fields in it are in space perspective from the vertex D according to the definition of Art. 1. D is the **pole** of the space perspective and π is the **plane of perspective.**

To define the space perspective completely we require another pair of points F_1, F_2 which may be taken anywhere on a line through D. The vanishing planes then contain the line at infinity in π and so are parallel to π, and constructions for corresponding points, planes and lines can be worked out by a simple generalisation of those of Art. 16.

If, however, we take the self-corresponding point E on an edge BC of the self-corresponding tetrahedron (but not at B or C) then every point of this edge is self-corresponding and every plane through the opposite edge AD is self-corresponding. It is then easily seen that the homographic fields in each of the self-corresponding planes ABC, DBC are fields in plane perspective. We may refer to this type of homography as **uniaxal.** To define it, we require another pair of corresponding points F_1, F_2, which must be taken in a plane through AD, but are otherwise arbitrary, save that they must not lie on AD itself or in the planes ABC, DBC. If two such points F_1, F_2 coincide at F, then every point of the plane λ through AD, in which they lie, is self-corresponding. Every plane of space meets λ in a self-corresponding line and BC in a self-corresponding point, and so is self-corresponding and the homography reduces to an identity.

If the plane ADF meet BC at X and F_1, F_2 coincide at F on one of AX or DX, say on AX, then there are three self-corresponding points A, F, X on AX and this line is wholly self-corresponding. Two lines AX, BC in the plane ABC are then wholly self-corresponding, so that the plane ABC is wholly self-corresponding. We then fall back on the case of space perspective.

But if F_1 and F_2 coincide at F on AD, we have AD wholly self-corresponding and we have the case of two non-intersecting lines or **axes** of homography, every point of which, and every plane through which, is self-corresponding. We may refer to this type of homography as **biaxal.** To define it we require another pair of corresponding points G_1, G_2, which must now be taken on a line meeting both axes at X, Y.

No further degeneration is now possible, for if G_1 and G_2 were now to coincide at G, the line XY would become wholly self-corresponding. The planes determined by XY, AD and by XY, BC would then be wholly self-corresponding and every plane (and therefore every point) of space would be self-corresponding.

269. Involutory space homographies. If in a space homography two distinct points correspond doubly, their join is a self-corresponding line, upon which the homography determines an involution whose double points are the only points of this line self-corresponding in the homography. If there be two such involutory self-corresponding lines a, b in the homography, which do not intersect, the double points of the involutions on a, b form a self-corresponding tetrahedron. If a, b intersect, then the point and plane ab are a self-corresponding point and self-corresponding plane, and Art. 168 shows that the homography determines, in this plane, a harmonic plane perspective. In either case, no conclusion can be drawn as to whether any other pair of distinct points, not lying on a, b or in the plane ab when a, b intersect, correspond doubly.

If, however, there be a third involutory self-corresponding line c, we have to consider the following cases :

I. a, b, c are all skew to one another. Let P_1, P_2 be two corresponding points on a, and let p_1, p_2 be the lines through P_1, P_2 which meet b, c ; p_1, p_2 are thus uniquely determined when P_1, P_2 are known, and they are corresponding lines, since b, c are self-corresponding. The cobasal reguli $[p_1]$, $[p_2]$ are homographic, and, in fact, form an involution, having double lines x, y, which are self-corresponding in the homography. On each of x, y are three distinct self-corresponding points, namely those in which x, or y, meets a, b, c. Hence every point on x or y is self-corresponding and the homography is a biaxal homography, with x, y as axes.

II. a intersects b, but c does not lie in the plane ab. If O is the point ab, α the plane ab, then O is self-corresponding. Let A, B be the other self-corresponding points on a, b and let $AB=x$. The homography determines in α a harmonic plane perspective in which O is the pole and x the axis of perspective. Now the point $c\alpha \equiv D$ must be self-corresponding and so is either at O or lies on x. If D is at O, a, b, and c are concurrent but not coplanar; we reserve this case for further consideration. If D lies on x, then, since the correspondence in α is a harmonic plane perspective, $OD \equiv d$ is an involutory self-corresponding line in the homography, which therefore determines, in the plane $cd=\beta$, another harmonic plane perspective, of which D is the pole, and a line y in β is the axis, where y passes through O and through the other self-corresponding point C on c. This point C is not in α, hence x, y are skew lines. Clearly every point of x, y is self-corresponding in the homography, which is thus again biaxal.

III. a, b, c are concurrent at O, but not coplanar. O is self-corresponding; let A, B, C be the other self-corresponding points on A, B, C. As before, the homography determines, in each of the three faces of the three-edge abc, a harmonic plane perspective. The three axes of perspective are the lines BC, CA, AB, every point of which is self-corresponding, whence it follows that every point of the plane ABC is self-corresponding and the homography reduces to a space perspective.

If now in either a biaxal homography, or a space perspective, a pair of distinct points P_1, P_2 correspond doubly, the homography will be involutory. For it is always possible to take arbitrarily two points A, B on one axis x of the biaxal homography, and two points C, D on the other axis y, such that of the five points A, B, C, D, P_1, no four are coplanar. And since P_1P_2 meets both AB and CD, then C cannot lie in the plane ABP_2 without also lying in the plane ABP_1, which we have just excluded; similarly for the other cases. Thus of A, B, C, D, P_2, no four are coplanar.

In like manner it is possible to take three arbitrary points A, B, C in the plane of perspective π of the space perspective in such a way that, D being the pole of perspective, no four of A, B, C, D, P_1 are coplanar, and therefore, using the fact that P_1P_2 passes through D, no four of A, B, C, D, P_2 are coplanar.

The homography is then determined by the transformation of

A, B, C, D, P_1 into A, B, C, D, P_2. But, since P_1, P_2 correspond doubly, if $P_2 = Q_1$, then $P_1 = Q_2$. The same homography is therefore determined by the correspondence which transforms A, B, C, D, Q_1 into A, B, C, D, Q_2, that is, A, B, C, D, P_2 into A, B, C, D, P_1. It is therefore identical with its reverse, that is, every pair of distinct corresponding points correspond doubly and the homography is involutory.

Since every pair of mates are harmonically separated by the double point of an involution, any two corresponding points in such an involutory homography are harmonically separated (i) by the axes when the homography is biaxal, (ii) by the pole and plane of perspective when the homography is a space perspective.

An involutory biaxal homography is usually referred to as a **skew involution** and an involutory space perspective as a **harmonic space perspective.**

These two are the only possible types of involutory space homography.

270. Any quadric can be transformed homographically into any other quadric. Let ψ_1 be a quadric, A_1, B_1, C_1 any three points on it, a_1, a_1' the generators through A_1, b_1, b_1' the generators through B_1. Let a_1b_1' be D_1, $a_1'b_1$ be E_1. These data define the quadric ψ_1 entirely. For, if through C_1 we draw a line c_1' meeting the two generators a_1, b_1 (unaccented generators belonging to the same system), then c_1' has three points of the quadric on it, and is then a generator of the accented system. We have thus three generators of the latter system and the quadric is determined.

On another quadric ψ_2 take in like manner three arbitrary points A_2, B_2, C_2 and the generators a_2, a_2' through A_2; b_2, b_2' through B_2, those of opposite systems meeting at $a_2b_2' = D_2$ and $a_2'b_2 = E_2$. These data define ψ_2 entirely.

Consider now the homography in which A_1, B_1, C_1, D_1, E_1 correspond to A_2, B_2, C_2, D_2 E_2. It transforms ψ_1 into a quadric ψ_2'—since a homographic transformation leaves order, class and degree unaltered—and ψ_2' passes through A_2, B_2, C_2 and has the line $A_2D_2 = a_2$ a generator and also $B_2D_2 = b_2'$, $A_2E_2 = a_2'$, $B_2E_2 = b_2$ are generators. But these data determine ψ_2, which therefore coincides with ψ_2', and corresponds homographically to ψ_1.

It follows that any property which is preserved by homography, such as the non-metrical properties of incidence, tangency and cross-ratio, holds for all quadrics if it can be proved for any (non-degenerate) quadric. It should, however, be borne in mind that degeneracy is also preserved by homography, so that we cannot argue in this way from properties of a degenerate quadric to those of the general quadric.

Further, any conic s_1 may be transformed homographically into any other conic s_2, with any three given points A_1, B_1, C_1 of s_1 corresponding to any three given points A_2, B_2, C_2 of s_2. We have already seen how to do this by a plane homography (Art. 166) between fields in different planes. It can, however, also be done by a single space homography as follows. Let the tangents to s_1 at A_1, B_1 meet at T_1, those to s_2 at A_2, B_2 meet at T_2. On any lines through T_1, T_2, not in the planes of s_1, s_2 respectively, take two pairs of points D_1, E_1 ; D_2, E_2. Then the homography in which A_1, B_1, C_1, D_1, E_1 correspond to A_2, B_2, C_2, D_2, E_2 transforms T_1 into T_2 and s_1 into a conic touching T_2A_2 at A_2, T_2B_2 at B_2 and passing through C_2, that is, into s_2.

Any conic may therefore be transformed homographically into the circle at infinity, and any quadric of which that conic is a plane section then becomes a sphere. The non-metrical properties of the general quadric may therefore be deduced from those of the sphere in the same way as those of the general conic are deduced from those of the circle.

271. Space correlation. In the same way as we construct a space homography we can construct a **space correlation** in which points correspond to planes and planes to points, straight lines as joins of two points corresponding to straight lines as meets of two planes. The equations of transformation are obtained from those of Art. 267 by interchanging x', y', z' with l', m', n'.

Note that, in any correlation, a surface-locus of any order corresponds to a surface-envelope of the same class, a curve of any degree to a developable of the same class, an axial pencil to a homographic range, a flat pencil to a homographic flat pencil, a regulus to a homographic regulus, a cone of the second order to a conic, and a quadric to a quadric.

As in Art. 173 we can consider incident points and planes in such a correlation. If a point $P_1 = R_2$ lies on its corresponding plane π_2, then if ρ_1 is the plane corresponding to R_2, since R_2 is a point of π_2, then ρ_1 passes through P_1, so that a point is incident, in whichever field we take it, and through every incident point pass its two corresponding planes. Similarly in every incident plane lie its two corresponding points.

In general, corresponding lines do not intersect; if, however, two corresponding lines x_1, x_2 intersect, the point and plane which they determine are an incident point and plane. For, let $P_1 = R_2$ be the intersection of two such lines. Since P_1 lies on x_1, its corresponding plane π_2 passes through x_2 and so must contain P_1, since x_2 contains it. Similarly ρ_1 which corresponds to R_2 must pass through x_1 and contains R_2. These planes, however, are not necessarily identical with the plane x_1x_2. Similarly if $\alpha_1 = \beta_2$ is the plane x_1x_2, then since α_1 passes through x_1, its corresponding point A_2 lies on x_2, and therefore in α_1, and the point B_1 corresponding to β_2 lies on x_1 and therefore in β_2. These points A_2, B_1 are not necessarily identical with the point x_1x_2.

If now x is any line in space, P_1 any point of it, π_2 its corresponding plane meeting x at P_2, the ranges $[P_1]$ $[P_2]$ are homographic. Being cobasal, they have two self-corresponding points U, V. If P_1 coincides with either U or V, it coincides with P_2 at that point and therefore lies in π_2. Every straight line has accordingly two incident points on it; hence the locus of incident points is a quadric ψ_1.

Similarly, if π_1 be now any plane through x, P_2 its corresponding point, the axial pencils $x[P_2]$ and $[\pi_1]$ are homographic. Their self-corresponding planes α, β are such that, when treated as planes of the field Φ_1, they contain their corresponding points in Φ_2. Thus through any line x two incident planes can be drawn, hence the envelope of incident planes is a quadric ψ_2.

If P be a point common to ψ_1, ψ_2, let g_1, h_1 be the two generators of ψ_1 through P. Since every point of g_1 lies on ψ_1, every plane through g_2 is tangent to ψ_2, so that g_2 is a generator of ψ_2; similarly h_2 is a generator of ψ_2. Now $P \equiv g_1h_1$ corresponds to the plane g_2h_2, and since P is an incident point it lies in this plane. But P is also a point of ψ_2 and so lies in at least one of g_2, h_2. If P lies on g_2, then if Q_1 be any other point of g_1, Q_1 is an incident point and lies in the corresponding plane κ_2 which passes through g_2; and these planes κ_2 are all different. But all the points Q_1 necessarily lie in the determinate plane g_1g_2; this requires that the points Q_1 must lie on g_2, that is g_1 and g_2 coincide. Hence the intersection of ψ_1, ψ_2 must consist entirely of common generators, so that, in general, it is a skew quadrilateral (Art. 266) and ψ_1, ψ_2 touch at the four vertices of this quadrilateral.

If, however, ψ_1 be a cone, then ψ_2 is a disc quadric, the envelope of the tangent planes to a conic s_2. In this case the generator g_1 of ψ_1 through P must correspond to a tangent g_2 to s_2. Here again the points of g_1 lie on their corresponding planes through g_2, which are all different, and as before g_1 coincides with g_2. The vertex of ψ_1 corresponds to the plane of ψ_2 and lies in it. This plane meets the cone ψ_1 in two generators g_1, which touch s_2 at two points. This case may be deduced from the more general one by making the four common generators coincide in pairs, members of a pair belonging to opposite systems.

As in Art. 173 we may speak of ψ_1, ψ_2 as the **base quadrics** of the space correlation.

If the base quadrics coincide in a single quadric ψ, then, by the above reasoning, every generator of one system on ψ is self-corresponding in the space correlation; and, in general, the generators of the other system correspond homographically with two self-corresponding members.

If, however, every generator of both systems on ψ be self-corresponding, then any tangent plane to ψ corresponds to its point of contact, and this is still the case if the two fields be interchanged. Hence if α_1, β_1 are the tangent planes to ψ through any line p_1, A_2, B_2 their points of contact with ψ, the line A_2B_2 corresponds doubly to p_1. Thus polar lines with respect to ψ correspond doubly in the correlation; and by considering intersecting lines it is seen that any point in either field corresponds doubly to its polar plane with respect to ψ in the other. We have now a transformation by reciprocal polars with respect to the base quadric ψ.

As a particular case the quadric ψ may be a sphere of centre O and radius of reciprocation a, which may be arbitrary. We then have point-reciprocation in three dimensions, a point P_1 corresponding to a plane π_2 perpendicular to OP_1 and meeting OP_1 at P_2 where $OP_1.OP_2=a^2$.

In such point-reciprocation the dihedral angle between two planes is equal to the angle subtended at O by their corresponding points; also two corresponding lines are perpendicular, and the angle between any two lines a_1, b_1 is equal to the dihedral angle between the planes Oa_2, Ob_2.

272. Inversion with regard to a sphere. Precisely as in Chapter XI we can define inversion with respect to a sphere centre O and radius a by taking the points P_1, P_2 of the last part of Art. 271 to correspond.

Such a transformation (as in the plane) is not a homography.

The following properties of inversion with regard to a sphere are easily established, following the lines of Chapter XI, and are set down below without proof, for reference.

Every sphere not passing through O inverts into a sphere not passing through O, and every sphere passing through O inverts into a plane; also any circle not passing through O inverts into a circle not passing through O, a circle in a plane through O inverts into a coplanar circle, and every circle passing through O into a coplanar straight line.

Every sphere through two inverse points is orthogonal to the sphere of inversion.

A sphere orthogonal to the sphere of inversion inverts into itself.

A set of spheres which intersect on a fixed plane (the common radical plane) have on their line of centres two limiting points which are point-spheres of the set, and invert with respect to either of these limiting points into concentric spheres.

Two corresponding surfaces or curves through inverse points P_1, P_2 are equally inclined to the line OP_1P_2, the normals to the two surfaces (or the tangents to the two curves) being coplanar with OP_1P_2, but not parallel to each other. Hence any small elementary solid inverts into an oppositely similar solid (*i.e.* with right and left interchanged), and the transformation is conformal.

If two inverse fields are simultaneously inverted with respect to any centre, they invert into inverse fields, their spheres of inversion inverting into one another.

273. The twelve-face eight-point. There exists, in three dimensions, an analogue to the harmonic property of the complete quadrangle and complete quadrilateral in a plane.

Let three pairs of planes α_1, α_2; β_1, β_2; γ_1, γ_2 be described to pass through the sides BC, CA, AB respectively of a triangle ABC lying in a plane δ. These three pairs of planes define a harmonic space perspective, of which the plane of perspective is δ and the pole is the point D of concurrence of the planes through BC, CA, AB harmonically conjugate to δ with respect to α_1, α_2; β_1, β_2; γ_1, γ_2 respectively. We shall denote the faces BCD, CAD, ABD of the tetrahedron $ABCD$ by α, β, γ.

In what follows p, q, r will denote any set of suffixes each of which is either 1, 2; p', q', r' will be the complementary suffixes to p, q, r, so that if $p=1$, $p'=2$ and conversely.

The points $\alpha_p\beta_q\gamma_r$ are 8 in number and form the **vertices** of a figure which may be called an **eight-point.** Since, in a space harmonic perspective, every pair of corresponding elements correspond doubly, lines such as $\beta_q\gamma_r$, $\beta_{q'}\gamma_{r'}$ correspond, and the plane containing them passes through D. Denote the plane $(\beta_1\gamma_1, \beta_2\gamma_2)$ by λ_1, $(\beta_1\gamma_2, \beta_2\gamma_1)$ by λ_2. The planes β_1, β_2, γ_1, γ_2 belong to the star vertex A and form in this star a complete four-face, of which λ_1, λ_2 and ABC are the diagonal planes, and λ_1, λ_2 meet in AD. By the harmonic property of the complete four-face, λ_1 and λ_2 are harmonically conjugate with respect to the planes DAC, DAB, *i.e.* β and γ. Similarly the planes $\mu_1 = (\gamma_1\alpha_1, \gamma_2\alpha_2)$, $\mu_2 = (\gamma_1\alpha_2, \gamma_2\alpha_1)$ pass through BD and are harmonically conjugate with respect to γ, α. Again the planes $\nu_1 = (\alpha_1\beta_1, \alpha_2\beta_2)$, $\nu_2 = (\alpha_1\beta_2, \alpha_2\beta_1)$ pass through CD and are harmonically conjugate with respect to α, β.

Moreover, every vertex $\alpha_p\beta_q\gamma_r$ lies on a straight line $\beta_q\gamma_r$ and therefore on a plane λ_n. There are four vertices in λ_1, namely $\alpha_1\beta_1\gamma_1$, $\alpha_1\beta_2\gamma_2$, $\alpha_2\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, and four vertices in λ_2, namely $\alpha_1\beta_1\gamma_2$, $\alpha_1\beta_2\gamma_1$, $\alpha_2\beta_1\gamma_2$, $\alpha_2\beta_2\gamma_1$.

A plane which contains four vertices of the eight-point will be termed a **face** of the eight-point, and two faces which contain between them all the eight vertices will be termed opposite faces.

It will be clear from what has gone before that λ_1, λ_2, and by symmetry also μ_1, μ_2 and ν_1, ν_2 are pairs of opposite faces. It is also immediately clear, from the original mode of construction of the vertices of the eight-point, that the pairs α_1, α_2 ; β_1, β_2 ; γ_1, γ_2 are likewise pairs of opposite faces. There are thus 12 faces, and for this reason we shall refer to the figure as a **twelve-face eight-point.** That there can be no more than 12 such faces can be seen as follows. There are 56 planes obtained by taking the 8 vertices 3 at a time ; of these each of the 12 faces contributes four coincident ones, making a total of 48. There remain 8 planes containing only 3 vertices ; such vertices are readily recognised to be of type $\alpha_p\beta_q\gamma_r$, $\alpha_{p'}\beta_{q'}\gamma_r$, $\alpha_{p'}\beta_q\gamma_{r'}$, and there are as many of them as there are vertices $\alpha_p\beta_q\gamma_r$, for they cannot repeat themselves, as if we take, for example, p', q' instead of p, q, the rule gives $\alpha_{p'}\beta_{q'}\gamma_r$, $\alpha_p\beta_q\gamma_r$, $\alpha_p\beta_{q'}\gamma_{r'}$, where the third vertex is new and a distinct plane is obtained. Accordingly all the 56 planes are accounted for, and there can be no additional faces.

It now appears that the properties of the eight-point are entirely symmetrical with respect to the tetrahedron $ABCD$. This tetrahedron will be referred to as the **diagonal tetrahedron** of the eight-point. Its vertices, faces and edges will be termed the **principal diagonal points, planes and lines** of the eight-point.

Accordingly we may define the eight-point by three pairs of opposite faces passing through three coplanar edges of the diagonal tetrahedron, *e.g.* α_1, α_2; μ_1, μ_2; ν_1, ν_2 through BC, BD, CD. These pairs will correspond in a harmonic space perspective of which A is the pole and α the plane of perspective.

But since every vertex lies in one or other of a pair of opposite faces, any three pairs of such faces will define the eight vertices.

Vertices of the type $\alpha_p\beta_q\gamma_r$, $\alpha_{p'}\beta_{q'}\gamma_{r'}$ will be called **complementary vertices**; the same applies to vertices of the type $\alpha_p\mu_q\nu_r$, $\alpha_{p'}\mu_{q'}\nu_{r'}$, and similarly in other cases. Lines joining complementary vertices will be called **edges** of the eight-point. An edge joining vertices $\alpha_p\beta_q\gamma_r$, $\alpha_{p'}\beta_{q'}\gamma_{r'}$, which are corresponding points in the plane perspective of pole D, passes through D, and there are clearly four such edges.

Similarly there are four edges through A (of the form $\alpha_p\mu_q\nu_r$, $\alpha_{p'}\mu_{q'}\nu_{r'}$), four through B and four through C, so that there are altogether 16 edges of the eight-point.

If we now consider two pairs of planes (α_1, α_2) (λ_1, λ_2) which pass through opposite edges BC, AD of the diagonal tetrahedron, each of the four lines $\alpha_p\lambda_q$ meets both BC and AD, and contains two vertices $\alpha_p\lambda_q\beta_r$ and $\alpha_p\lambda_q\beta_{r'}$, (β_1, β_2) being any other pair of opposite planes. These are not complementary vertices, and the lines $\alpha_p\lambda_q$ will be termed **diagonals** of the eight-point. Every diagonal meets two opposite edges of the diagonal tetrahedron; four meet the same two such edges, so that there are 12 in all.

Moreover, since there are only 28 joins of 8 points two at a time, the 16 edges and 12 diagonals exhaust the possible combinations.

Through any edge $\beta_1\gamma_1$ pass three faces $\beta_1, \gamma_1, \lambda_2$, no two of which are opposite.

Through any diagonal $\alpha_1\lambda_1$ pass only the two faces α_1, λ_1.

Through any vertex $\alpha_1\beta_1\gamma_1$ pass the six faces $\alpha_1, \beta_1, \gamma_1, \lambda_1, \mu_1, \nu_1$ no two of which are opposite, the four edges which join this vertex to A, B, C, D and the three diagonals which are the lines through this vertex meeting pairs of opposite principal diagonal lines.

In any face α_1 lie four edges $\alpha_1\beta_1$, $\alpha_1\beta_2$, $\alpha_1\gamma_1$, $\alpha_1\gamma_2$. The edges $\alpha_1\mu_1$, $\alpha_1\mu_2$, $\alpha_1\nu_1$, $\alpha_1\nu_2$ are identical with $\alpha_1\gamma_1$, $\alpha_1\gamma_2$, $\alpha_1\beta_1$, $\alpha_1\beta_2$ respectively. In the same face α_1 lie the two diagonals $\alpha_1\lambda_1$ and $\alpha_1\lambda_2$.

274. The twelve-point eight-face. Reciprocating the arguments of the preceding Article, we can construct a figure with eight faces and twelve vertices, through each one of which pass four faces.

A repetition of the proof would be tedious, although the student is advised to go through it carefully for himself, using the method of translation explained in Art. 57 and adapted to the case of three dimensions. The results will here be briefly indicated.

On three edges $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ of a three-edge of vertex D pairs of points A_1, A_2; B_1, B_2; C_1, C_2 are taken. From these eight planes $A_pB_qC_r$ are obtained, which form the eight faces. The triangles $A_1B_1C_1$, $A_2B_2C_2$ define a harmonic space perspective with D as pole and a plane δ as plane of perspective, meeting $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ at A, B, C. The diagonal points (other than D) of the quadrangles, such as $B_1B_2C_1C_2$, give pairs of points L_1, L_2; M_1, M_2; N_1, N_2 on BC, CA, AB, that is, on $\alpha\delta$, $\beta\delta$, $\gamma\delta$ respectively. We have twelve vertices A_1, A_2; B_1, B_2; C_1, C_2; L_1, L_2; M_1, M_2; N_1, N_2 opposite in pairs, any three pairs determining the eight-face. These are connected in groups of three by harmonic space perspectives having A, B, C, D and α, β, γ, δ as poles and planes of perspective, respectively.

Meets of complementary faces $A_pB_qC_r$, $A_{p'}B_{q'}C_{r'}$ give 16 edges. Each edge lies in a principal diagonal plane, that is, a face of the diagonal tetrahedron $\alpha\beta\gamma\delta$. In each principal diagonal plane lie four edges, *e.g.* the edge given by $A_1B_2C_1$, $A_2B_1C_2$ lies in δ and contains the points $L_2M_1N_2$. Every edge has three vertices on it. Joins of vertices such as A_1L_1, A_1L_2, A_2L_1, A_2L_2, which lie on opposite principal diagonal lines give the 12 diagonals.

Through every vertex pass 4 faces, 4 edges and 2 diagonals. In every face lie 4 edges, 3 diagonals and 6 vertices, which are the intersections of the face with the 6 principal diagonal lines.

Every principal diagonal plane contains 4 edges and 6 vertices, forming a complete quadrilateral of which the three principal diagonal lines in that plane form the diagonal triangle.

Through every edge and every diagonal pass only two faces.

275. Harmonic properties of the eight-point and eight-face. If we consider a diagonal of the eight-point, say $\alpha_1\lambda_1$ (*see* Art. 273), meeting two opposite principal diagonal lines BC, AD, the two vertices on it are determined by a pair of opposite faces such as β_1, β_2, passing through AC. But β_1, β_2 are harmonically conjugate with respect to the two principal diagonal planes CAB, CAD, that is, δ, β passing through their intersection CA. These four planes meet $\alpha_1\lambda_1$ at the two vertices on it and at its intersections with the lines BC, AD, which lie in δ, β respectively. Thus (1) **every diagonal is harmonically divided by the principal diagonal lines which it intersects.**

We have already seen that (2) **every pair of opposite faces are harmonically conjugate with respect to the principal diagonal planes through their intersection.**

Further, since every face of the eight-point contains one principal diagonal line, the six faces through a vertex are the planes through that vertex and the six edges of the diagonal tetrahedron. Hence (3) **they are the six faces of the complete four-edge formed by the four edges of the eight-point through the given vertex, and the three diagonals are the three diagonal lines of this four-edge.**

Consider now any edge $\beta_1\gamma_1$ passing through the principal diagonal point A. The vertices on it lie on opposite faces α_1, α_2, which pass through BC and are harmonically conjugate with respect to the planes $BCA=\delta$ and $BCD=\alpha$. Thus the two vertices, in which the planes α_1, α_2 of the axial pencil $(\alpha\alpha_1\delta\alpha_2)$ meet $\beta_1\gamma_1$, are harmonically conjugate with respect to the points where α, δ meet $\beta_1\gamma_1$, that is (4) **any edge is harmonically divided by the principal diagonal point on it and the opposite principal diagonal plane.**

By reciprocation, or directly, we obtain the corresponding harmonic properties of the twelve-point eight-face, as follows, corresponding propositions being correspondingly numbered.

(1) **The two faces through a diagonal are harmonically conjugate with respect to the two planes containing that diagonal and the principal diagonal lines which intersect it.**

(2) **Every pair of opposite vertices are harmonically conjugate with respect to the principal diagonal points on their join.**

(3) **The six vertices in a face are the six vertices of the complete quadrilateral formed by the four edges in that face, and the three diagonals in that face are the diagonals of this quadrilateral.**

(4) **The two faces through an edge are harmonically divided by the principal diagonal plane through that edge, and the plane joining the edge to the opposite principal diagonal point.**

276. Associated eight-point and eight-face. It follows from property (3) of the twelve-point eight-face in the last Article that a diagonal A_1L_1 lying in a face $A_1B_1C_1$ is harmonically divided at points P_1, P_2 by the diagonals B_1M_1, C_1N_1 lying in that plane. But A_1L_1 also lies in the face $A_1B_2C_2$, and so is harmonically divided by the diagonals B_2M_2, C_2N_2 lying in that plane.

Now the faces through B_1M_1 are $A_1B_1C_1$, $A_2B_1C_2$, and the faces through A_1L_1 are $A_1B_1C_1$, $A_1B_2C_2$. Hence A_1L_1, B_1M_1 meet on the intersection of the planes $A_2B_1C_2$, $A_1B_2C_2$, that is, on the diagonal C_2N_2. Therefore the non-coplanar diagonals A_1L_1, B_1M_1, C_2N_2 are concurrent at P_1, and similarly A_1L_1, B_2M_2, C_1N_1 are concurrent at P_2.

We see then that the diagonals of the eight-face are concurrent in threes at points such as P_1, P_2 which are not vertices of the eight-face. On each diagonal there are two such points, making 24 in all, but in this sum each point is counted three times over, so that there are eight points of type P_1, P_2.

We will now prove that these eight points form a twelve-face eight-point.

The points P_1, P_2 on A_1L_1 are the intersections of A_1L_1 with C_2N_2 and C_1N_1. Similarly we have points P_3, P_4 on A_2L_1 which are the intersections of A_2L_1 with C_2N_1 and C_1N_2. But on each of the four diagonals C_2N_2, C_1N_1, C_2N_1, C_1N_2 there is a second point of the same class. Let these be Q_1, Q_2, Q_3, Q_4.

Now P_1, P_2, P_3, P_4 lie in the plane $A_1A_2L_1$, which we may term λ_1 and which passes through the principal diagonal line AD. The points N_1, N_2 lie in the principal diagonal line AB and therefore in the principal diagonal plane $ABD=\gamma$. Similarly C_1, C_2 lie on CD and therefore in the plane $ACD=\beta$. Let the plane ADQ_1 be λ_2. Since C_2N_2 is harmonically divided by P_1, Q_1, we have $AD\{C_2P_1N_2Q_1\}=-1=\{\beta\lambda_1\gamma\lambda_2\}$: λ_2 is therefore the plane through AD harmonically conjugate to λ_1 with respect to β, γ. Proceeding in like manner with the ranges $(C_1P_2N_1Q_2)$ $(C_2P_3N_1Q_3)$ $(C_1P_4N_2Q_4)$ we find that Q_1, Q_2, Q_3, Q_4 all lie on λ_2. The eight points P, Q thus lie on two planes through AD, harmonically conjugate with regard to the principal diagonal planes through AD.

By symmetry the same holds good of any other edge and it follows that these eight points are vertices of a twelve-face eight-point, having the same diagonals, and the same diagonal tetrahedron, as the original twelve-point eight-face.

We shall say that an eight-point and eight-face related in this manner are **associated.**

We might have arrived at the same result by proving that the diagonals of the eight-point are coplanar in threes, and lie in eight planes forming the associated eight-face.

277. Eight-point or eight-face with given diagonal tetrahedron. We will now show that a twelve-face eight-point can always be uniquely constructed if the diagonal tetrahedron $ABCD$ and one vertex P are arbitrarily given. Join PA to meet BCD at E_1 and on PA take P_1 harmonically conjugate to P with regard to A, E_1. In like manner construct P_2, P_3, P_4.

Through P draw a line to meet the two opposite principal diagonals BC, AD at F_1, G_1 and on F_1G_1 take Q_1 harmonically conjugate to P with respect to F_1, G_1. In like manner, from the opposite principal diagonals (CA, BD) (AB, CD) construct Q_2, Q_3.

It is clear that, since PP_2, PP_3, PQ_1 all intersect BC, the four points P, P_2, P_3, Q_1 lie in a plane α_1 through BC. Now if α_2 is the plane through BC harmonically conjugate to α_1 with regard to ABC, BCD, then since $\{APE_1P_1\} = -1$ and on this transversal A, P, E_1 lie on ABC, α_1, BCD respectively, P_1 lies on α_2. Similarly P_4 lies on α_2. Further, if F_2, G_2 are points on AC, BD such that F_2PG_2 is a straight line and $\{F_2PG_2Q_2\} = -1$, then since F_2, P, G_2 lie on ABC, α_1, BCD respectively, Q_2 lies on α_2. Similarly Q_3 lies on α_2.

The eight points therefore lie in fours on two planes α_1, α_2 through BC, harmonically conjugate with regard to BCA, BCD. Similarly they lie in fours on two such planes β_1, β_2 through CA and on two planes γ_1, γ_2 through AB, etc.

They form accordingly a twelve-face eight-point having $ABCD$ for its diagonal tetrahedron.

In like manner a twelve-point eight-face is uniquely determined from its diagonal tetrahedron and one face, which can be arbitrarily given.

Examples

1. Prove that no new points can be obtained by repeating the construction, starting from any of the seven points P_1, P_2, P_3, P_4, Q_1, Q_2, Q_3.

2. Prove that any twelve-face eight-point can be homographically transformed into any other, and also that any twelve-point eight-face can be homographically transformed into any other.

3. Show that a twelve-face eight-point is entirely determined from five arbitrary points in space, no four of which are coplanar, three of these being principal diagonal points and two being complementary vertices of the eight-point.

4. Prove that any twelve-face eight-point and its associated twelve-point eight-face can be homographically transformed into a cube and its inscribed regular octahedron.

EXAMPLES XV

1. Five skew lines a, b, c, d, e have two common transversals l, m; and the transversals n, n' from one point E of e to the pairs of lines a, b and c, d respectively are coplanar with e. Show that the quadric bases of the reguli (a, b, e) (c, d, e) have the same tangent plane at every point of e.

2. The locus of the vertex of a cone of the second order inscribed in a given skew hexagon is a quadric.

3. Through the points where the planes of an axial pencil meet a straight line are drawn perpendiculars to these planes. Show that these perpendiculars lie in a hyperbolic paraboloid.

4. If a wrap of tangent planes to a cone of the second order be homographic with an axial pencil whose base does not pass through the vertex of the cone, but which is such that the wrap and the axial pencil have one self-corresponding plane, their product is a quadric.

5. A range of points on a conic is homographic with a range on a straight line not coplanar with the conic but meeting the conic at A. If A be a self-corresponding point show that the joins of corresponding points of the two ranges lie on a quadric.

6. Prove that the focal axis of a plane section of a right circular cone is equal to the part of any generating line intercepted between its points of contact with the focal spheres, and that the perpendicular axis is a mean proportional between the diameters of the focal spheres.

7. Prove that the latus rectum of a plane section of a right circular cone is proportional to the perpendicular distance of the plane of section from the vertex of the cone.

8. In any plane section of a right circular cone, prove that the absolute length of the non-focal semi-axis is a mean proportional between the distances of the vertices on the focal axis of the section from the axis of the cone.

9. Prove that the generators of a quadric through the extremities of conjugate diameters of a plane section of the quadric intersect on two planes parallel to the plane of the section.

10. A parallelogram $ABCD$ is inscribed in a quadric ψ: prove that any plane parallel to that of $ABCD$ meets the tangent planes to ψ at A, B, C, D in the four sides of a parallelogram.

11. Show that if two quadrics have a common generator the generators of the other system in each quadric, which intersect on their common twisted cubic, form homographic reguli.

12. Prove that through any point P of space a quadric can be drawn containing a given twisted cubic and a given chord of it.

Show that through P one chord, and one only, of a given twisted cubic can be drawn.

13. Prove that a unique quadric surface can be drawn through a given twisted cubic t and two given points not on the same chord of t.

14. Prove that every twisted cubic can be obtained as the intersection of two cones of the second order, whose vertices A and B lie on the cubic and that the rays joining A and B to any four points P, Q, R, S of the cubic are equi-anharmonic in the conical pencils of the second order formed by the generators of the cones.

15. Two given homographic star fields have different vertices. Show that, if two corresponding lines of the stars intersect, their intersection lies on a given twisted cubic.

16. Two given homographic plane fields lie in different planes. Show that, if two corresponding lines intersect, the plane through them envelops a developable of the third class.

17. **Homographic ranges of the third order on a twisted cubic being defined as ranges which are projected from any chord of the cubic by homographic axial pencils, with a corresponding definition for an involution on the cubic, prove that the joins of pairs of mates (P, P') of an involution on a twisted cubic are the generators of a regulus.**

Show also that the involution (P, P') belongs to the biaxal space involution whose axes are those generators of the complementary regulus which pass through the double points of the involution (P, P').

18. If A, B', C, A', B, C' are six fixed points of a twisted cubic, P any point of the curve, prove that the transversals from P to the pairs of lines (AB', $A'B$) (BC', $B'C$) (CA', $C'A$) lie in a plane π; and show that, as P describes the curve, π turns about a fixed chord of the cubic, which joins the self-corresponding points of the homographic ranges of the third order defined on the cubic by the triads (A, B, C), (A', B', C').

19. Prove that a developable of the third class is entirely determined by any six of its tangent planes, and show how to construct (a) the tangent to the cuspidal edge, (b) the point of contact of this tangent with the cuspidal edge, corresponding to any one of these tangent planes.

20. Show that the common tangent planes to two quadrics with a common generator envelop a developable of the third class, and that, through each of the generators of one system in each quadric, two planes of the developable pass, but through each of the generators of the other system there passes only one plane of the developable. Show further that, in general, an arbitrary line does not lie in a plane of the developable and cannot in any case lie in more than two.

21. Prove that the tangent planes of a developable of the third class meet a given tangent plane of the developable in a pencil of the second order; and hence that every such developable can be generated by the common tangent planes to two conics which have a common tangent line.

22. Two plane fields in planes π_1, π_2 are in space perspective. Show that cross-joins A_2B_1, A_1B_2 meet on a fixed plane γ passing through $\pi_1\pi_2$, and that the two fields define a harmonic space perspective, in which γ is the plane of perspective.

23. α, β are two planes; a, b two non-coplanar lines in space which both meet $\alpha\beta$. Show that if P_1, P_2 be points of α, β respectively such that P_1P_2 meets a and b, the correspondence between the plane fields $[P_1]$, $[P_2]$ is homographic.

24. Prove that, if two harmonic space perspectives have the pole of either lying on the plane of perspective of the other, and a field ϕ corresponds to ϕ_1 in one perspective and to ϕ_2 in the other, ϕ_1 and ϕ_2 are related fields in a harmonic biaxal homography.

25. Prove that, of the six faces through each of two complementary vertices of a twelve-face eight-point, three are common and three are opposite.

26. Two complete quadrilaterals have in common two vertices on a diagonal and also the two vertices of the diagonal triangle on this diagonal. Show that their remaining vertices form a twelve-face eight-point.

27. Discuss the problem of constructing a tetrahedron of which the vertex O shall be in a given plane α, the faces that meet at O shall pass through given lines l, m, n and the base ABC shall be an equilateral triangle in a given plane β. How many solutions are there?

28. Show that the intersection of a plane π_1 of a star vertex O_1 with the corresponding line p_2 of a reciprocal star vertex O_2 lies on a quadric Q passing through O_1, O_2, and the intersection of p_1 and π_2 lies on the same quadric. Prove that Q meets any plane α in the locus of incident points in the reciprocal fields determined on α by the stars.

29. If ϕ, ϕ' be two reciprocal fields in different planes, the plane joining a point of ϕ to its corresponding line of ϕ' envelops a quadric Q, the tangent cone to which from any point O is the envelope of incident planes in the two reciprocal stars of vertex O by which ϕ, ϕ' are projected from O.

30. Show that, given three skew lines a, b, c, a definite line d exists in space, which is harmonically conjugate to a with respect to b, c in a regulus.

Show further that two polar lines with respect to a quadric ψ determine, in each system of generators of ψ, an involution, and find the double lines of these involutions.

CHAPTER XVI

FURTHER PROPERTIES OF QUADRICS

278. Self-polar tetrahedron. Let P be any point, π its polar plane with respect to a quadric, R a point of π, and ρ its polar plane, which passes through P. Also let S be a point of $\pi\rho$, and σ its polar plane, which passes through both P and R and meets $\pi\rho$ at a point T. Then T lies on π, ρ, σ: its polar plane τ is PRS. A tetrahedron such as $PRST$ is said to be self-polar with regard to the quadric. Each vertex is the pole of the opposite face. Any two of its vertices, or any two of its faces, or any two of its edges, are conjugate with regard to the quadric and any two of its opposite edges are polar lines.

Any three-edge $\alpha\beta\gamma$ whose faces are mutually conjugate for the quadric ψ is termed **self-conjugate** for ψ. If δ is the polar plane of the point $\alpha\beta\gamma$, any two of the four planes α, β, γ, δ are conjugate for ψ and the tetrahedron $\alpha\beta\gamma\delta$ is self-polar. The triangle in which the three-edge $\alpha\beta\gamma$ meets the plane δ is self-polar for the section of ψ by δ; and the three-edge is therefore self-polar for the tangent cone to ψ from its vertex.

Again any triangle ABC whose vertices are mutually conjugate for ψ is termed a **self-conjugate** triangle for ψ; it is self-polar for the conic in which the plane ABC meets ψ; and if D is the pole of the plane ABC with regard to ψ, the tetrahedron $ABCD$ is self-polar.

It should be noted that two edges of a self-conjugate three-edge, or sides of a self-conjugate triangle, are conjugate, but not polar, lines with respect to the quadric. The polar plane of any vertex of a self-conjugate triangle contains the opposite side and the pole of any face of a self-conjugate three-edge lies in the opposite edge.

In general a self-conjugate three-edge does not meet an arbitrary plane in a self-conjugate triangle, unless that plane is the polar plane of the vertex of the three-edge. In like manner the lines joining the vertices of a self-conjugate triangle to any point do not form a self-conjugate three-edge, unless the point is the pole of the plane of the triangle.

If, however, the quadric ψ degenerates into a cone of vertex O, then O is the pole of any plane π, not passing through O. If now a is any line through O, α its conjugate diametral plane with respect to the cone (Art. 241), then every line which meets a and α is harmonically divided by the cone. In this case any point of a may be regarded as the pole of α. If abc be a self-polar three-edge of vertex O for the cone, then a, b, c will meet π at points A, B, C which are poles of α, β, γ respectively, so that A, B, C, O are mutually conjugate in pairs; thus $OABC$ is a self-polar tetrahedron for the cone, and every such tetrahedron must have O for a vertex.

In this case the self-polar three-edge through O meets every plane π in a self-conjugate triangle, and every self-conjugate triangle in a plane not passing through O is projected from O by a self-polar three-edge.

If a twelve-face eight-point is inscribed in a quadric ψ, then its diagonal tetrahedron is self-polar for the quadric. For reverting to the notation of Art. 273, if we take the four lines $\beta_1\gamma_1$, $\beta_1\gamma_2$, $\beta_2\gamma_1$, $\beta_2\gamma_2$ through A, each of these is an edge of the eight-point; thus $\beta_1\gamma_1$, for example, contains the two vertices $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_1\gamma_1$. A is the point $\delta\beta_1\gamma_1$ and the point where the plane BCD meets $\beta_1\gamma_1$ is $\alpha\beta_1\gamma_1$. But α_1, α_2 are harmonically conjugate with respect to α, δ by the property of the eight-point. Hence the point $\alpha\beta_1\gamma_1$ is harmonically conjugate to A with respect to $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_1\gamma_1$ which are two points of the quadric on a chord through A. Thus $\alpha\beta_1\gamma_1$ is a point on the polar plane of A. Similarly $\alpha\beta_1\gamma_2$, $\alpha\beta_2\gamma_1$, $\alpha\beta_2\gamma_2$ all lie on the polar plane of A, which must accordingly be α. Similarly β is the polar plane of B, γ of C and δ of D. $ABCD$ is therefore a self-polar tetrahedron for ψ.

It follows from Art. 277 that, if a self-polar tetrahedron for ψ be given, and also one point P on the quadric, seven other points can be at once constructed, which lie on the quadric and form a twelve-face eight-point. For, in the construction of Art. 277, the same points P_1, P_2, P_3, P_4, Q_1, Q_2, Q_3 are obtained as points of the quadric, if we make use of the properties that each face of $ABCD$ is the polar plane of the opposite vertex and that every chord of the quadric intersecting two opposite edges of $ABCD$ (which are polar lines) is harmonically divided by these lines.

Examples

1. Show how to construct a quadric, given a self-polar tetrahedron, and a given point and plane as pole and polar.

2. Show that an infinite number of self-polar tetrahedra for a given quadric can be constructed, having two given conjugate points P, Q for vertices, and that their other vertices form an involution.

3. Prove that, if through an edge $\gamma\delta$ of a self-polar tetrahedron $\alpha\beta\gamma\delta$ any plane π is drawn, meeting the opposite edge $\alpha\beta$ at E, the three planes through E, namely α, β, π, form a self-conjugate three-edge.

4. Prove that, if on an edge CD of a self-polar tetrahedron $ABCD$ a point P is taken, the three points ABP form a self-conjugate triangle.

5. Prove that, if a quadric contains seven vertices of a twelve-face eight-point, it must also pass through the eighth vertex.

6. Prove that if a quadric touches seven faces of a twelve-point eight-face, it must touch the eighth face.

7. Show that, in general, one quadric can be described through three given points of space and having a given tetrahedron for a self-polar tetrahedron.

8. If three quadrics have a common self-polar tetrahedron, their eight intersections form a twelve-face eight-point, and their eight common tangent planes form a twelve-point eight-face, and these have the same diagonal tetrahedron.

279. Centre, principal axes and planes. The pole of the plane at infinity is termed the **centre** of the quadric. The ellipsoid and hyperboloids have their centre at a finite distance. The paraboloids have no accessible centre. Lines and planes through the centre are **diameters** and **diametral planes** respectively. As in the case of the conic all diameters are bisected at the centre, when this is accessible. Diameters of a paraboloid are parallel to a fixed direction.

The polar plane π of a point P^∞ at infinity passes through the pole of the plane at infinity and so is a diametral plane, which is conjugate to all lines p passing through P^∞. Such parallel lines p determine chords of the quadric which are harmonically divided by π and P^∞. Hence **chords parallel to a given direction are bisected by their conjugate diametral plane.**

If d^∞ is a line at infinity, its polar line d' passes through the pole of the plane at infinity and so is a diameter. Any point C' of d' is conjugate to every point of d^∞ for the quadric; hence C' is the centre of the section of the quadric by the plane $C'd^\infty$, which plane is conjugate to d'. Thus **the locus of centres of sections of the quadric by a system of parallel planes is the diameter conjugate to those planes.**

A self-conjugate three-edge whose vertex is the centre forms a system of three conjugate diameters, and its faces a system of three conjugate diametral planes. Any one of its edges is the diameter conjugate to the opposite face.

In the case of an ellipsoid or hyperboloid with centre O, any three mutually conjugate diameters form three edges of a self-polar tetrahedron $OA^\infty B^\infty C^\infty$. Chords parallel to a diameter OA^∞ are bisected by its conjugate diametral plane $OB^\infty C^\infty$; and the locus of centres of sections by planes parallel to a diametral plane $OB^\infty C^\infty$ is the diameter OA^∞ conjugate to that plane.

The plane at infinity meets the quadric in a conic k^∞. In general k^∞ and $\odot$ have one and only one common self-polar triangle $I^\infty J^\infty K^\infty$. The lines OI^∞, OJ^∞, OK^∞ are the only set of three mutually perpendicular conjugate diameters of the quadric, and are termed the **principal axes** of the quadric. Each is perpendicular to its conjugate diametral plane; these three planes are called the **principal planes** of the quadric. A principal plane is a plane of symmetry for the quadric, since every chord parallel to an axis is bisected by the perpendicular principal plane.

In the case of a paraboloid k^∞ degenerates into a line-pair, namely the two generators of the quadric in the plane at infinity. In this case there is still a common self-polar triangle $I^\infty J^\infty K^\infty$, but one vertex K^∞ is now the double point of the line-pair, that is, the point of contact of the plane at infinity with the paraboloid, and I^∞, J^∞ determine two perpendicular directions in a plane perpendicular to the direction of K^∞. There are now two accessible principal planes, namely those which bisect chords passing through I^∞, J^∞; these are the polar planes of I^∞ J^∞ and their intersection is the only accessible principal axis of the paraboloid. The polar plane of K^∞ is the plane at infinity itself; the other two axes are therefore at infinity. As before, the accessible principal planes are planes of symmetry for the paraboloid, which therefore has only two-plane symmetry. In any set of three mutually conjugate diameters of a paraboloid, two are in the plane at infinity; and the plane at infinity must be one of any set of three conjugate diametral planes.

Returning now to the case of the general quadric, with accessible centre O, if k^∞ and $\odot$ touch at two points A^∞, B^∞, they have any number of common self-polar triangles, namely those having for vertices the common pole K^∞ of $A^\infty B^\infty$ with respect to k^∞ and $\odot$ and any pair of conjugate points I^∞, J^∞ on $A^\infty B^\infty$. There is accordingly a determinate axis OK^∞, but any two rectangular lines OI^∞, OJ^∞ in a plane perpendicular to OK^∞ are also axes. Further, any section of the quadric by a plane through $I^\infty J^\infty$ contains A^∞, B^∞. Since these are circular points, this section is a circle, of which the centre lies on the axis OK^∞. Thus sections of the quadric by planes perpendicular to OK^∞ are circles, whose centres lie on OK^∞. The quadric is therefore a *surface of revolution*, and, in the case of an ellipsoid, is termed a **spheroid.**

Finally, k^∞ and $\odot$ may coincide, in which case every triangle $I^\infty J^\infty K^\infty$ self-polar for $\odot$ gives a set of axes. The quadric then contains $\odot$ and reduces to a sphere, since every plane section of it is then a circle. Any set of conjugate diameters of a sphere is trirectangular and thus a set of principal axes.

The reader can easily trace for himself the modifications of the above necessary for the (real) paraboloid of revolution. It may, however, be worth while to point out that A^∞, B^∞ are necessarily conjugate imaginary points, for K^∞ is here real, and the planes perpendicular to the direction of K^∞ are also real, and therefore must meet $\odot$ in two points A^∞, B^∞ which are conjugate imaginary. Now k^∞ cannot then be a *real* line-pair, for $A^\infty B^\infty$, joining conjugate imaginary points, would be real, and its intersections A^∞, B^∞ with k^∞ would also be real, which is not the case. Thus k^∞ is an imaginary line-pair and a paraboloid of revolution must be an elliptic paraboloid (Art. 257).

280. Asymptotic cone and planes of circular section. The tangent cone to the quadric ψ from the centre touches the quadric along the conic k^∞ in which ψ meets the plane at infinity. This cone is termed the **asymptotic cone** of the quadric. Since it meets the plane at infinity in the same conic that the quadric does, it has the same axes and principal planes as the quadric.

Also any plane π meets the quadric and its asymptotic cone in two conics q and t which have their points at infinity common and so are similar (Art. 235).

Hence planes parallel to the cyclic planes of the asymptotic cone cut the quadric in circles (Art. 250). The tangent planes parallel to the planes of circular section meet the quadric in point-circles. Their points of contact are called **umbilics** of the quadric. Since there are six cyclic planes, of which two are real, there are twelve umbilics, of which four are real; and they lie in fours in the three principal planes.

An ellipsoid, since it has no real points at infinity, has clearly no real asymptotic cone. On the other hand a hyperboloid has a real asymptotic cone. We have seen in Art. 257 that if the tangent planes at infinity meet the quadric in real lines, then the hyperboloid has real generators; this requires that the hyperboloid should lie *outside* its asymptotic cone, since the tangent plane to a cone of the second order, like the tangent to a conic, has no real points inside the cone. The surface consists of a single sheet entirely surrounding the cone. Hence the name hyperboloid of one sheet.

If, on the other hand, the hyperboloid has no real generators, the tangent planes to the asymptotic cone do not intersect the hyperboloid in real generators. The quadric then consists of two symmetrical portions, one inside each of the two opposite half-cones which make up the asymptotic cone. Hence the name hyperboloid of two sheets (*see* Art. 257).

In the case of a quadric of revolution, since k^∞ has double contact with $\odot$, the asymptotic cone is a right circular cone (Art. 245) whose axis is the axis of revolution.

The asymptotic cone of a sphere is, of course, the spherical cone through the centre.

In the case of a paraboloid k^∞ becomes a line-pair, whose members are u^∞, v^∞, say. There is here no accessible centre, the asymptotic cone degenerates into the envelope of the tangent planes through the generators u^∞, v^∞, that is, into the line-pair k^∞ itself.

If U^∞, V^∞ are the points in which a plane γ perpendicular to the accessible axis meets u^∞, v^∞, respectively, let x, y be the second generators of the paraboloid through U^∞, V^∞. Then x, y belong to opposite systems, so that u^∞, v^∞, x, y form a skew quadrilateral of generators. The tangent planes xu^∞, yv^∞ at U^∞. V^∞, are termed the **asymptotic planes** of the paraboloid. They intersect in the polar line of $U^\infty V^\infty$, that is, in the accessible axis. Also they are harmonic with respect to the two accessible principal planes, since a pair of conjugate planes through the axis are harmonic with respect to the two tangent planes through the axis. Thus the accessible principal planes bisect the dihedral angles between the asymptotic planes.

Since lines parallel to the accessible axis already meet the paraboloid in one real point at infinity on that axis, they meet the surface at only one other real point. Thus every real paraboloid is a one-sheeted surface.

The point xy is termed the **vertex** of the paraboloid and the plane xy is the tangent plane at the vertex and perpendicular to the accessible axis.

A plane parallel to this axis meets the paraboloid in a section which touches the plane at infinity at its point of contact with the paraboloid. Such sections are therefore parabolas whose axes are parallel to the accessible axis of the paraboloid.

A plane perpendicular to the axis meets the paraboloid in a conic passing through U^∞, V^∞.

If k^∞ is a real line-pair, the paraboloid is a hyperbolic paraboloid; the points U^∞, V^∞, the asymptotic planes, and the lines x, y are all real, and the sections by planes γ perpendicular to the axis are hyperbolas having parallel asymptotes. The line-pair xy divides these hyperbolas into two sets, which lie on opposite sides of the plane xy and in supplementary dihedral angles formed by the asymptotic planes, so that the surface is saddle-shaped.

If k^∞ is an imaginary line-pair with a real double point, the points U^∞, V^∞, the asymptotic planes and the lines x, y are conjugate imaginary. The paraboloid is then an elliptic paraboloid, which lies entirely on one side of the plane xy; all its sections by planes perpendicular to the axis are ellipses.

Examples

1. A tangent plane to the asymptotic cone meets the quadric in parallel generators belonging to opposite systems.

2. Prove that two intersecting generators of a quadric are asymptotes of the section of the asymptotic cone by the plane containing the generators.

281. Common self-polar tetrahedron of two quadrics. Let ψ_1, ψ_2 be two quadrics, P_1 any point, π its polar plane with respect to ψ_1, P_2 the pole of π with respect to ψ_2. The relation between the fields $[P_1]$, $[P_2]$ is algebraic and one-one. Moreover, if P_1 describes a line p_1, π revolves about the polar line p of p_1 with respect to ψ_1, and P_2 describes the polar line p_2 of p with respect to ψ_2. Straight lines therefore correspond to straight lines and the space fields $[P_1]$, $[P_2]$ are homographic. This homography has, in general, four, and only four, self-corresponding points A, B, C, D, each one of which has the same polar plane, α, β, γ, δ respectively, with respect to ψ_1, ψ_2. Since $\beta\gamma\delta$ has BCD for its polar plane with respect to both ψ_1, ψ_2, the vertices of the tetrahedron $\alpha\beta\gamma\delta$ must be identical with the points A, B, C, D so that the tetrahedron

$ABCD$ is self-polar for both quadrics, and is the only such tetrahedron.

If we now transform the fields $[P_1]$, $[P_2]$, by a homography, so that B, C, D become trirectangular points at infinity, the quadrics become coaxial quadrics, with a common centre at the point corresponding to A. All non-metrical properties of two quadrics can therefore be derived from those of two coaxial quadrics.

The fields $[P_1]$, $[P_2]$, however, may have more than four self-corresponding points; this happens when they are related by a uniaxal or biaxal homography, or by a space perspective.

If the homography between $[P_1]$, $[P_2]$ is uniaxal, and x is the axis, then x is a line of self-corresponding points. If x meet ψ_1 at A and B, the tangent planes α, β to ψ_1 at A, B are polar planes of A, B respectively with respect to both ψ_1 and ψ_2, so that A, B lie in their polar planes with respect to ψ_2 and α, β touch ψ_2 at A, B. Hence ψ_1, ψ_2 have double contact and intersect along two conics. If C, D are the two self-corresponding points of the homography not on x, U any point of x, the plane UCD has the same pole V for both quadrics. V is therefore a self-corresponding point of the homography, which is not in general coincident with U, C or D and therefore is a point of x, which is harmonically conjugate to U with respect to A, B. We have then an infinite number of common self-polar tetrahedra $UVCD$.

If the homography between $[P_1]$, $[P_2]$ is biaxal, then, if the axes x, y meet ψ_1 at A, B; C, D respectively, we have, as before, that ψ_1, ψ_2 touch at the four points A, B, C, D, in which case their intersection is a skew quadrilateral of generators. It is then easily shown that if (S, T), (U, V) are any pairs of harmonic conjugates with respect to (A, B), (C, D) respectively, the tetrahedron $STUV$ is self-polar for both quadrics.

If the homography between $[P_1]$, $[P_2]$ is a plane perspective, it is proved as before that the quadrics touch all along a conic k in which they are both cut by the plane of perspective. The vertex of the common tangent cone is a self-corresponding point of $[P_1]$ $[P_2]$ not in the plane of perspective, and so is the pole O of perspective. If ABC is any self-polar triangle for the conic k, $OABC$ is a common self-polar tetrahedron for both quadrics.

So far we have considered those special cases where there are more than four self-corresponding points of $[P_1]$ $[P_2]$. But other special cases also arise when two or more of the four self-corresponding points A, B, C, D coincide, in which case there are less than four points which have the same polar plane for both quadrics.

First let A and B coincide along a line u. Then A lies in its polar plane $BCD=ACD$ with respect to both quadrics. These accordingly touch at A, the plane ACD being the common tangent plane: CD and u are common polar lines for ψ_1, ψ_2.

Let, further, C and D also coincide along a line v. Then A lies in its polar plane Av and C lies in its polar plane Cu, with respect to both quadrics. Hence ψ_1 and ψ_2 touch at A and C, the tangent planes Av, Cu passing through AC. Thus AC is a generator of both quadrics, the remainder of their intersection being a twisted cubic. The lines u, v are polar lines for both quadrics.

If A, B, C coincide, the sides BC, CA, AB approaching a, b, c as limits, then A is the pole, with regard to both quadrics, of three planes aD, bD, cD, all passing through AD, but generally distinct.

If A, B, C, D coincide, then A is the pole, with regard to both quadrics, of four planes α, β, γ, δ through A, which are generally distinct.

This clearly implies that the relation between pole and polar plane is no longer unique for either quadric; hence ψ_1, ψ_2 must be degenerate quadrics. Moreover, the homography between the fields $[P_1]$, $[P_2]$ would cease to be determinate.

Such cases must therefore be excluded from consideration here, unless the above three (or four) planes happen to coincide. When this is so, the point A will be found to be a point where the quadrics have contact of higher order; the investigation of this will, however, be omitted.

282. Pencil and range of quadrics. The ten coefficients in the equation of a quadric through eight given points must satisfy eight linear equations; they can therefore in general be expressed as homogeneous linear functions of two arbitrary parameters λ_1, λ_2. The equation of any quadric through the eight points is therefore of the form

$$\lambda_1 S_1 + \lambda_2 S_2 = 0,$$

where S_1, S_2 are definite expressions of the second degree in the coordinates. Every quadric through the eight points therefore passes through the twisted quartic curve of intersection of the quadrics

$$S_1 = 0, \quad S_2 = 0.$$

Hence the set of quadrics through eight given points contains in general a determinate twisted quartic.

It should be noted that the above reasoning assumes that the eight given points are of sufficiently general position for the eight linear equations satisfied by the coefficients to be linearly independent. It will be shown later (*see* Art. 292) that all quadrics through seven given points pass also through an eighth fixed point.

Such a set of quadrics is termed a **pencil** of quadrics and their common twisted quartic will be referred to as the **base** of the pencil.

Through any point of space, not lying on the above quartic, one quadric of the pencil passes.

Reciprocating the above results we see that the quadrics which touch eight given planes touch a determinate developable of the fourth class. They are said to form a **range** of quadrics, and the developable is termed the **base** of the range.

One quadric of the range touches any given plane, which is not a tangent plane of the base developable.

283. Properties of a pencil of quadrics. A pencil of quadrics determines an involution on any straight line. Two quadrics of the pencil touch this straight line at the two double points of this involution. These two double points are conjugate for all the quadrics of the pencil and may be said to be **conjugate points for the pencil.** Since two pairs of mates determine the double points of an involution, it is clear that if two points P, P' are conjugate for two quadrics ψ_1, ψ_2 of a pencil, they are conjugate for the pencil. If then π_1, π_2 are the polar planes of P with respect to ψ_1, ψ_2, their meet $p' = \pi_1, \pi_2$ is the locus of points P' conjugate to P for the pencil, and we may call this the line **conjugate to** P **for the pencil.**

The conics in which the quadrics of a pencil meet any plane form a pencil of conics passing through the four points in which the twisted quartic which defines the pencil meets this plane. Three of the quadrics of the pencil therefore meet the plane in line-pairs, that is, they touch the plane at the centres of the line-pairs. These centres are the vertices of the common self-polar triangle of the conics in which the pencil of quadrics meets the plane. Hence:

In every plane there is one triangle self-conjugate with regard to all the quadrics of a pencil. Its vertices are the points of contact of the three quadrics of the pencil which touch the plane.

A pencil of quadrics has, in general, one, and only one, common self-polar tetrahedron. For let $ABCD$ be the common self-polar tetrahedron of two of the quadrics of the pencil, ψ_1 and ψ_2, then any pair of its vertices, such as A, B, are conjugate for every quadric ψ of the pencil, and therefore the tetrahedron is self-polar for every such quadric ψ.

Cases of exception arise, as in Art. 281, whenever there are one or more points where all the quadrics of the pencil touch. The detailed discussion of these cases will be omitted.

Any ray through a vertex, say A, of the common self-polar tetrahedron of a pencil of quadrics, which meets the quartic base at a point P, meets it again at a point P'. For, if AP meet the opposite face of the tetrahedron at L, AP meets both quadrics again at the point harmonically conjugate to P with respect to A and L. This point, therefore, lies in the quartic base. Hence the four points P, P', R, R' in which any plane through A meets the quartic lie in two pairs (P, P') (R, R') on rays through A. The quartic is therefore projected from A by a cone which has two generators in any plane through A, that is, a cone of the second order. A similar result holds for the other vertices B, C, D of the tetrahedron.

Hence, in general, **four of the quadrics of a pencil are cones, whose vertices are the vertices of the common self-polar tetrahedron of the pencil.**

The above assumes that the common self-polar tetrahedron is both unique and proper.

Of the cases of exception we will only consider one, namely that when the quartic base breaks up into two conics. In this case the common self-polar tetrahedron has two distinct vertices A and B through which pass two cones of the pencil and an edge CD every point of which has the same polar plane for every quadric of the pencil. This edge is the double line of the plane-pair formed by the planes of the two conics. The two remaining cones of the pencil therefore coalesce with this plane pair.

If the quartic base reduces to a skew quadrilateral, no proper cone can be drawn through the intersection, but there are then two plane-pairs, of which the diagonals of the skew quadrilateral are the double lines.

If we take a point P on the twisted quartic common to a pencil of quadrics, the twelve-face eight-point of which P is a vertex and $ABCD$ the diagonal tetrahedron is inscribed in every quadric of the pencil, and therefore in the twisted quartic. By varying P we see that an infinite number of such eight-points can be inscribed in any twisted quartic which is the intersection of two quadrics. In view of the degenerate character of the self-polar tetrahedron for two quadrics with a common generator the same result does not generally hold when the quartic breaks up into a line and a twisted cubic.

An important particular case of a pencil of quadrics occurs when two of the quadrics are spheres; the quartic base then breaks up into the circle at infinity and an accessible circle c (which may be real or imaginary) lying in a real plane α, which is termed the **radical plane** of the two spheres.

Clearly every quadric of the pencil contains $\odot$ and so is a sphere. Also each passes through the circle c; and any two have α for their radical plane.

Such spheres provide an analogue of coaxal circles in a plane, and may be called coaxal spheres. Their centres lie on the line through the centre of c perpendicular to its plane. Also any plane meets such a pencil of spheres in a set of coaxal circles.

It is easily proved that the tangents from any point on the radical plane to all the spheres of the pencil are equal. There are two cones of the system, which are spherical cones and therefore point-spheres lying on the line of centres, giving the limiting points of the system. All spheres through the limiting points meet every sphere of the pencil orthogonally and have their centres on the radical plane. These spheres, however, do not form a pencil of quadrics. Their centres all lie on the common radical plane of the original pencil, and they provide another type of three-dimensional generalisation of coaxal circles, namely spheres through two points.

284. Polar quadric of a line for a pencil of quadrics. It is clear that the polar plane π of P with regard to a quadric ψ of a pencil passes through the line p' which is conjugate to P for the pencil.

If now π be a given plane through p', ψ is uniquely determined. For, if P do not lie in the base quartic, let Q be any point of the base quartic; join PQ meeting π at S and let R be harmonically conjugate to Q with respect to P, S. Then R is a point of ψ and, in general, determines ψ uniquely. If P lies on the base quartic, p' is the tangent line at P to the base quartic, π is the tangent plane at P to the quadric ψ. A line in π, other than p', gives a point of ψ ultimately coincident with P, but not lying on the base quartic, and this also determines ψ. Hence, when π is known, ψ is determined, and conversely. Thus the pencil of quadrics $[\psi]$ is homographic (*cf.* Art. 208) with the axial pencil $[\pi]$.

If now u is any straight line, P_1, P_2 two points of it, π_1, π_2 their polar planes with regard to a quadric ψ of the pencil, we have $[\pi_1]\barwedge[\psi]\barwedge[\pi_2]$.

Hence $\pi_1\pi_2$ generates a regulus in a quadric ϕ, in which the axes p_1', p_2' of $[\pi_1]$, $[\pi_2]$ are generators of the other system.

Now $\pi_1\pi_2$ is the polar line of u with respect to ψ and is independent of the choice of P_1, P_2.

Thus the polar lines of a given line u with respect to the quadrics of a pencil form a regulus of generators of a quadric ϕ, which is the **polar quadric** of u with regard to the pencil.

Since p_1' has been shown to be a generator of ϕ of the other system, and P_1 can be taken arbitrarily on u, the conjugate lines to points of u with respect to the pencil form the second set of generators of the polar quadric.

Finally any point Q of the polar quadric of u lies on a line conjugate to a point P of u with regard to the pencil, and so is conjugate for the pencil to a point P of u, and also lies on the polar line of u with regard to some quadric ψ of the pencil, so that it is the pole of some plane through u with regard to ψ.

Thus the polar quadric is also : (i) the locus of points conjugate to points of u for the pencil ; (ii) the locus of poles of planes through u for quadrics of the pencil.

The polar quadric ϕ of u passes through the vertices A, B, C, D of the common self-polar tetrahedron of the pencil. For if the plane BCD meet u at U, then A and U are conjugate for the pencil, so that A must lie on ϕ ; and similarly for B, C and D.

ϕ also passes through each of the two points of u which are conjugate for the pencil $[\psi]$, since each of these is conjugate to a point of u, namely the other point of the pair.

Note also that ϕ meets any plane λ through u in the eleven-point conic of u for the pencil of conics in which $[\psi]$ meets λ.

285. Polar cubic of a plane for a pencil of quadrics. Let π be any plane, A, B, C any three non-collinear points on it, α, β, γ the polar planes of A, B, C with regard to a quadric ψ of a pencil.

By Art. 284, α, β, γ correspond in homographic axial pencils, whose axes are the lines a, b, c conjugate to A, B, C for the pencil. Thus the locus of $\alpha\beta\gamma\equiv P$, that is of the pole of $ABC\equiv\pi$, when ψ is varied, is a twisted cubic, of which a, b, c are chords. This cubic will be termed the **polar cubic** of π with respect to the pencil of quadrics. It passes through the points of contact of the three quadrics of the pencil which touch π, and also through the vertices of the common self-polar tetrahedron of the pencil, for, if D be any such vertex, D is the pole of any arbitrary plane and therefore of π, with respect to the cone of the pencil which has D for vertex.

Since the pole of π lies on the polar line of any line in π, it follows that the polar cubic of π lies in the polar quadric of any line in π.

In particular, if π is the plane ω^{∞} at infinity, the locus of the centres of the quadrics of a pencil is a twisted cubic passing through the vertices of the common self-polar tetrahedron and through the three points at infinity where ω^{∞} touches the three paraboloids of the pencil.

If the centre-locus meet a quadric ψ of the pencil at P, the tangent plane π to ψ at P meets ω^{∞} in a line i^{∞}. Since π is the polar plane of P for ψ, and ω^{∞} is its polar plane for some other quadric of the pencil, $i^{\infty}=\omega^{\infty}\pi$ is the line conjugate to P for the pencil.

If now one quadric of the pencil is a sphere σ, of centre O, P is conjugate to all the points of i^{∞} for σ. Hence the polar line of i^{∞} for σ is OP and consequently OP is perpendicular to all planes through i^{∞}, and thus to π. Therefore OP is the normal to ψ at P.

But it has been shown (Art. 266) that a twisted cubic meets a quadric at six points. There are accordingly six points P and six normals OP which can be drawn from O to the quadric ψ.

The centre-locus of the pencil defined by ψ and any sphere σ of centre O passes through the feet of the six normals from O, through O itself, through the centre C of ψ, through the four vertices of the self-polar tetrahedron of σ and ψ and through the points of contact X^{∞}, Y^{∞}, Z^{∞} of the paraboloids of the pencil with ω^{∞}.

But the last three points $X^{\infty}Y^{\infty}Z^{\infty}$ form the common self-polar triangle of $\odot$ and the conic k^{∞} in which ψ meets ω^{∞}. They thus correspond to three rectangular directions, mutually conjugate for ψ, that is, to the directions of the principal axes of ψ.

Thus the asymptotes of the twisted cubic are parallel to the principal axes of ψ.

The properties of this twisted cubic are thus analogous to those of the hyperbola of Apollonius (Art. 222) and it may be termed the **cubic of Apollonius** for the point O and the quadric ψ.

EXAMPLE

The six normals from a point to a quadric lie on a cone of the second order, of which three generators are parallel to the axes, and which contains the centre of the quadric.

286. Properties of a range of quadrics. The properties of a range of quadrics are immediately derivable from those of a pencil of quadrics by reciprocation. We will note the following:

The tangent cones from any point P to the quadrics of a range form a system touching four planes of a star vertex P.

Through any point P three quadrics of the range can be made to pass, the tangent planes to which at P form a three-edge self-conjugate for all the quadrics of the range.

A range of quadrics determines an involution of pairs of tangent planes through any straight line not in a common tangent plane, the double planes of the involution being the tangent planes to the two quadrics which touch the line.

Two planes which are conjugate for each of two quadrics of a range are conjugate for every quadric of the range, and the tetrahedron self-polar for two quadrics of the range is self-polar for every quadric of the range.

To any general plane π corresponds a line p' through which pass all planes conjugate to π for the quadrics of the range and which is also the locus of poles of π for the quadrics of the range.

Taking π at infinity the locus of centres of the quadrics of a range is a straight line.

The surface generated by the lines p' corresponding to planes π through a given line p is a quadric touching the four faces of the tetrahedron self-polar for all the quadrics of the range.

The poles of any two given planes with respect to a variable quadric of a range correspond, in general, in two ranges of the first order, which are homographic with each other and with the range of quadrics.

Bearing in mind that a cone reciprocates into a conic we see, reciprocating the property of Art. 283, that in general:

Four of the quadrics of a range are disc quadrics, whose planes are the faces of the common self-polar tetrahedron of the range.

287. Confocal quadrics. Consider the range determined by any quadric ψ_0 (not a paraboloid) and the circle $\odot$ at infinity (a degenerate case of a quadric). There are three conics of this range, besides the circle at infinity. Their planes α, β, γ and the plane at infinity form the self-polar tetrahedron of the range: α, β, γ are therefore three conjugate diametral planes of any quadric ψ of the range. They form a self-conjugate three-edge for every quadric of the range, in particular for $\odot$. Hence they are faces of a tri-rectangular three-edge, so that α, β, γ are principal planes of every quadric of the range.

Hence the quadrics ψ of such a range are concentric and coaxial. There are three conics of the range lying each in one of the three common principal planes.

These conics are called the **focal conics** of ψ: every point of them is called a **focus** of ψ.

The quadrics ψ are said to form a **confocal** system.

Let F be any point of a focal conic. Then the tangent cones from F to the quadrics of the confocal system form by Art. 286 a system of cones touching four fixed planes through F. Now consider the tangent cone to a conic from any point in its plane. This tangent cone (treated as an envelope) reduces to the two tangents from the point to the conic. Hence the tangent cone from F to the focal conic consists of two coincident tangents to this conic at F. The four fixed planes therefore consist of the two tangent planes to any cone of the system through the tangent line to the focal conic at F, each such tangent plane being *doubled*, that is, its line of contact being given. Hence every cone of the system touches two fixed planes through F along given lines through F in these planes, or the tangent cones from F to the system of confocals have double contact. But one of these tangent cones is the tangent cone to $\odot$, that is, it is the spherical cone through F. The tangent cones from F to the system of confocals have therefore double contact with the spherical cone; that is, they are right circular cones.

Foci of a quadric are thus points, the tangent cones from which to the quadric are right circular.

Through every point P three quadrics ψ_1, ψ_2, ψ_3 of a confocal system can be drawn. The three tangent planes π_1, π_2, π_3 form a three-edge self-conjugate for the range, and therefore for $\odot$. Hence they are mutually perpendicular, and the quadrics are orthogonal at all their points of intersection.

The line p', which is the locus of the poles of π_1 for the confocal quadrics, contains the pole of π_1 for $\odot$ and so is perpendicular to

π_1. It must also pass through the pole of π_1 for ψ_1, that is, through the point P. Hence **the locus of the poles of a fixed plane π_1 with respect to a set of confocal quadrics is the normal to the confocal which touches π_1, at its point of contact with π_1.**

EXAMPLES

1. Prove that if ψ is the locus of lines conjugate to planes π through a line p for a range of quadrics, the generators of ψ belonging to the complementary system are the polar lines of p for the quadrics of the range.

2. Prove that a single focal conic defines a family of confocal quadrics.

3. Show that, if t is the tangent at P to the intersection of two confocal quadrics ψ_1, ψ_2, the tangent planes through t to any other confocal quadric of the system are equally inclined to the tangent planes at P to ψ_1, ψ_2.

4. If the quadric ψ_0 of the above Article be a paraboloid, prove that the range consists entirely of paraboloids, and that there are only two focal conics, instead of three.

288. Lines of curvature on a quadric. A **line of curvature** on any surface is defined as a curve on the surface such that the normals to the surface at the points of this curve generate a developable. This is sometimes expressed by saying that the normals to the surface at any two consecutive points intersect.

If the normals to the surface at two points P, Q intersect, and Q approaches P in such a manner that the normal at Q always intersects the normal at P, the tangent at P to the locus of Q gives the direction at P of a line of curvature on the surface.

If the surface be a quadric ψ, the two generators PA, PB through P intersect the generators QA, QB through Q at A, B. The normals at P and Q are perpendicular to the tangent planes PAB, QAB and therefore perpendicular to AB. These normals intersect if, and only if, they are coplanar and so lie in a plane perpendicular to AB. The necessary and sufficient condition for this is that PQ is perpendicular to AB. Since PQ and AB are polar lines for ψ, the condition for the normals at Q, P to intersect is that PQ should be at right angles to its polar line.

When Q comes into coincidence with P along a curve satisfying this condition for every position of Q, the tangent t at P to this curve is perpendicular to its polar line t'. Since t lies in the tangent plane at P, and passes through P, so does its polar line t'; and t, t' are harmonic conjugates with respect to the two generators through P. Since t, t' are at right angles they bisect the angles between the generators.

Hence there are two lines of curvature through P, touching t, t' respectively at P. **The lines of curvature on ψ therefore form two orthogonal systems; the tangent to a line of curvature at any point is a bisector of the angles between the generators through that point.**

Note that, since the tangent plane at an umbilic (Art. 280) meets the quadric in a point-circle, if the point P above is an umbilic, the generators through P are circular lines, and any tangent line through P is perpendicular to its polar line. Thus an infinite number of lines of curvature, lying in all possible directions on the surface, pass through an umbilic.

Let now ψ_1, ψ_2 be the two quadrics confocal with ψ through P, and π, π_1, π_2 the tangent planes at P to ψ, ψ_1, ψ_2 respectively. The three planes π, π_1, π_2 form a three-edge self-conjugate for the confocal quadrics. Therefore the pole L of π_1 for ψ lies on $\pi\pi_2$; also the pole of π for ψ is P. The polar line of $\pi\pi_1$ for ψ is therefore $PL \equiv \pi\pi_2$. But π, π_1, π_2 are orthogonal (Art. 287); hence $\pi\pi_1$ and $\pi\pi_2$ are perpendicular polar lines for ψ, and therefore tangents at P to the lines of curvature through P on ψ. Since this is true of every point P on the intersection of ψ with ψ_1 (or ψ_2), it follows that **the lines of curvature on a quadric ψ are the intersection of ψ with the quadrics of the confocal system to which ψ belongs.**

289. Principal radii of curvature. The result of the first part of Art. 288 may also be obtained in a different manner.

Let O be a point of a quadric ψ; cut the quadric by a plane π parallel to the tangent plane at O. The section is a conic k, which is often spoken of as the **relative indicatrix** of O, which the student should be careful not to confuse with the spherical indicatrix mentioned in Art. 255.

Let ON be the normal at O, meeting π at N. Through N four normals can be drawn to the conic k, of which let the feet be P_1, P_2, P_3, P_4. At P_1 the tangent to the conic k is perpendicular to NP_1, and, since it lies in π, it is also perpendicular to ON. Therefore it is perpendicular to the plane ONP_1. Now the tangent to k at P_1 is also a tangent to the quadric at P_1, and a plane perpendicular to it must contain the normal to the quadric at P_1. Accordingly the normal at P_1 to the quadric lies in the plane ONP_1 and so must intersect ON. Similarly the normals at P_2, P_3, P_4 intersect ON. Hence, if π is made to move up to the tangent plane at O, the loci of P_1, P_2, P_3, P_4 touch at O the lines of curvature through O.

Now the centre C of k is the point where the diameter of ψ through O meets π, and the ratio $CN : ON$ is therefore constant, if π remains parallel to the tangent plane at O. On the other hand, if P is any point of k, the angle NOP approaches a right angle as N moves up to O, so that the ratio $ON : NP$ approaches zero. Accordingly the ratio $CN : NP$ approaches zero, that is, N approximates to the centre of the indicatrix. Thus, as we approach the limit, P_1, P_2, P_3, P_4 approach the feet of the normals to k from its centre, and the tangents to the lines of curvature at O are parallel to the axes of the indicatrix.

Since parallel sections of a quadric are similar and similarly situated, the asymptotes of k are parallel to the generators of ψ through O, and the axes of k are parallel to the bisectors of the angles between these generators, confirming the result obtained in Art. 288.

If we now consider the circle in the plane ONP_1 which touches the quadric at O and passes through P_1, its radius R_1 is given by $2ON.R_1 = OP_1{}^2$. In the limit, when the plane π approaches the tangent plane at O, this circle has three-point contact with the normal section of the quadric containing the tangent to the line of curvature which touches the locus of P_1. Calling ρ_1 the radius of this circle, ρ_1 is the limiting value of R_1.

Similarly if P_2 describes a curve touching the other line of curvature at O, so that P_1, P_2 approach different axes of the indicatrix, the radius of curvature ρ_2 at O of the normal section touching the second line of curvature is the limiting value of R_2, where $2.ON.R_2 = OP_2{}^2$.

These two normal sections are termed the **principal normal sections** of ψ at O; the centres K_1, K_2 of their circles of curvature at O are the **principal centres of curvature** of ψ at O; ρ_1, ρ_2 are the **principal radii of curvature** at O.

When the principal radii of curvature ρ_1, ρ_2 are known, the radius of curvature ρ of any other normal section, whose plane makes an angle ϕ with the principal normal section affected by the suffix 1, is easily obtained. For if this plane meet the indicatrix k at U, and R is the radius of the circle in this plane touching the quadric at O and passing through U, then $2.ON.R = OU^2 = ON^2 + NU^2$; and ρ is the limit of R. When the indicatrix is taken so near to the tangent plane at O that we may take N to coincide with the centre C of the indicatrix without sensible error, then, approximately

$$2.ON.\rho = CU^2; \quad 2.ON.\rho_1 = CA^2; \quad 2.ON.\rho_2 = CB^2,$$

where CA, CB are semi-axes of the indicatrix k, and if M be the foot of the perpendicular from U on CA, then $CM = CU\cos\phi$, $MU = CU\sin\phi$, to the same approximation.

Also we have (Art. 76)

$$\frac{CM^2}{CA^2} + \frac{MU^2}{CB^2} = 1$$

whence
$$\frac{1}{CU^2} = \frac{\cos^2\phi}{CA^2} + \frac{\sin^2\phi}{CB^2}.$$

Multiplying by $2.ON$ and proceeding to the limit, we obtain

$$\frac{1}{\rho} = \frac{\cos^2\phi}{\rho_1} + \frac{\sin^2\phi}{\rho_2}.$$

This formula shows that ρ_1 and ρ_2 are the maximum and minimum values of the radius of curvature of a normal section at O.

To justify the above procedure it should be noted that, although all the terms in the above approximate equations vanish in the limit, the ratio of the terms neglected to those which actually appear also tends to zero, so that the final formula is not approximate but exact.

We note that $\rho_1 : \rho_2$ is in the ratio of the squares of the semi-axes of the relative indicatrix k when π approaches the tangent at O; but since all such conics k are similar and similarly situated, the ratio in question is the ratio of the semi-axes of any relative indicatrix for all parallel positions of π.

If now a conic be constructed in the tangent plane at O, which is similar and similarly situated to any relative indicatrix, but with its centre at O and on such a scale that the squares of its semi-axes are equal to the corresponding principal radii of curvature ρ_1, ρ_2, this will be called the **absolute indicatrix** of O. If now the plane NOU above meet the absolute indicatrix at P, we find that

$$\frac{1}{OP^2} = \frac{\cos^2\phi}{\rho_1} + \frac{\sin^2\phi}{\rho_2} = \frac{1}{\rho},$$

so that $\rho = OP^2$ and is given by the square of the corresponding radius-vector of the absolute indicatrix.

290. Curvature of oblique sections. Meunier's Theorem. Let P be any point of a surface ψ (which need not here be restricted to be a quadric), PT any tangent line, α and β two planes through PT meeting the surface in curves h, k, which must touch PT at P; then, if R, S be any two points on h, k respectively, a sphere σ can be uniquely described touching PT at P and passing through R and S, since its centre C is determined by the meet of the perpendicular to the plane PRS through the circumcentre of the triangle PRS and the plane through P perpendicular to PT.

The planes α, β meet the sphere σ in circles c, d respectively, which touch PT at P and pass through R, S respectively, so that c touches h at P and meets it again at R, and d touches k at P and meets it again at S.

If now R, S coincide with P, σ becomes a sphere having four-point contact with the surface ψ at P. Also the circles c, d become the circles of curvature of h, k at P. The centre C of σ will then be on the normal to ψ at P.

Let α be taken to contain this normal, so that h is a normal section of ψ through PT; c is then a great circle of σ; if β makes an angle θ with α, d is a small circle of the sphere, touching c and inclined to c at an angle θ. The spherical centre of d is therefore at an angular distance θ from the spherical pole of c, and the spherical radius of d is $\frac{\pi}{2}-\theta$, so that its actual radius is $R\sin\left(\frac{\pi}{2}-\theta\right)$ or $R\cos\theta$, where R is the radius of the sphere, that is, of c.

Hence, **if the radius of curvature at P of a normal section h of a surface ψ is R, the radius of curvature of an oblique section touching h at P and inclined to the normal section at an angle θ is $R\cos\theta$. This is known as Meunier's Theorem.**

It is clear that the circles of curvature of all plane sections through PT must lie on the same sphere σ, since this sphere is entirely determined by the radius of curvature of the normal section through PT, and is independent of the choice of β.

But it should be noticed that, to different tangent lines PT will correspond different radii of curvature of normal sections (*see* Art. 289) and therefore different spheres σ. Thus, whereas in the plane we have only one circle having three-point contact with a given curve at a given point, there are an infinity of spheres having four-point contact with a given surface at a given point.

These are determined from the data that they have a common tangent plane at P (giving three coincident points) and, in addition, pass through another point R which coincides with P in a specified plane.

Meunier's Theorem enables us to see easily that the osculating plane of a line of curvature is not, in general, the normal plane to the surface of which it is a line of curvature. For if ψ_1, ψ_2 are two confocal quadrics, their intersection is a line of curvature on each of them. Taking a point P on this line of curvature, the corresponding principal radii of curvature of ψ_1, ψ_2 will be ρ_1, ρ_2 say. If now ρ is the radius of curvature of the line of curvature, considered as a twisted curve (Art. 253), and the osculating plane of this curve at P make an angle θ with the tangent plane π_1 to ψ_1 at P, $\rho = \rho_2 \cos\theta$, $\rho = \rho_1 \sin\theta$, by Meunier's Theorem, so that

$$\frac{1}{\rho^2} = \frac{1}{\rho_1^{\,2}} + \frac{1}{\rho_2^{\,2}}$$

and $$\tan\theta = \rho_2/\rho_1.$$

In general ρ_1 and ρ_2 are neither zero nor infinite, so that θ is neither zero nor a right angle.

291. Quadrics of curvature. If P be any point of a quadric ψ, another quadric ψ' can be drawn through P and eight other specified points. If two of the other eight points lie on ψ they may be made to come into coincidence with P in different directions; in the limit ψ and ψ' will then have the same tangent plane at P. If now three more of the eight points lie on ψ and are brought into coincidence with P in three different directions, the common normal planes through each of these directions meet ψ, ψ' in conics having the same radius of curvature ρ at P. The absolute indicatrices of P for ψ and ψ' have then a common centre and three other common points, and therefore must coincide entirely. Thus the quadrics ψ and ψ' have the same directions of principal curvature at P and the same curvature in every normal section through P. By Meunier's Theorem they have also the same curvature in every oblique section through P.

Since the sections for which the radius of curvature is infinite lie in the same planes, the quadrics must have the two generators through P common, so that the remainder of their intersection is a conic. The plane of this conic must pass through P, since the plane joining P to any two other common points Q, R (not lying in the tangent plane at P) meets the quadrics in two conics passing through Q, R and having three-point contact at P, so that the two conics coincide entirely.

A quadric ψ' which has contact of this kind with any surface σ will be termed a **quadric of curvature** for σ at the point P of contact. The directions at P of the lines of curvature on ψ' are the directions at P of the lines of curvature of σ and the principal radii of curvature of σ at P are those of ψ'. The curvature properties of σ at P are therefore identical with those of any quadric of curvature of σ at P. Since six of the nine points which determine ψ' have been made to coincide with P, a quadric of curvature at P can, in general, be made to pass through three other given points; in particular it may be made to touch a given plane at a given point, so that there is one paraboloid of curvature at P with its axis in any prescribed direction.

292. Net of quadrics. If a quadric passes through seven given points, we can show as in Art. 282 that its equation may be put into the form

$$\lambda_1 S_1 + \lambda_2 S_2 + \lambda_3 S_3 = 0,$$

S_1, S_2, S_3 being given expressions of the second degree in the co-ordinates and λ_1, λ_2, λ_3 arbitrary parameters. This quadric passes through the intersections of the three quadrics

$$S_1 = 0, \quad S_2 = 0, \quad S_3 = 0,$$

that is, quadrics satisfying such a condition pass through eight fixed points. Thus, in addition to the seven given points, there is an eighth fixed point, which is determined by the seven first, and through which the quadrics pass.

Such a set of eight points is termed a set of eight **associated points.**

The quadrics through seven given points are said to form a **net** of quadrics of which the given points are base points.

It should be noticed, however, that every quadric through seven given points on a twisted cubic curve passes through the curve, since, in general, a twisted cubic cannot meet a quadric in more than six points unless it lies entirely in the quadric (Art. 266). In this case the quadrics through the seven points form a net having the cubic for base curve.

The quadrics of a net which pass through another given point P_1 form a pencil, and so have a twisted quartic in common.

Consider then two pairs of quadrics of the net (ψ_1, ψ_1') and (ψ_2, ψ_2'), not belonging to the same pencil. Let P_1 be a point on the intersection of (ψ_1, ψ_1') other than the eight associated points, the base points of the net. Then the quadrics of the net, which pass through P_1, contain the intersection of ψ_1 and ψ_1'. Similarly if P_2 be a point on the intersection of ψ_2 and ψ_2' the quadrics of the net which pass through P_2 contain the intersection of ψ_2 and ψ_2'. Therefore the quadric of the net which passes through both P_1 and P_2 contains the intersections of ψ_1, ψ_1' and of ψ_2, ψ_2'.

We deduce that if four quadrics ψ_1, ψ_2, ψ_3, ψ_4 be such that the intersection of ψ_1, ψ_2 and that of ψ_3, ψ_4 lie on a quadric ψ the same is true however we choose the two pairs out of the four quadrics.

For consider the net defined by the quadrics ψ_1, ψ_2, ψ_3. Any quadric through the intersection of two quadrics of the net is a quadric of the net. Therefore ψ is a quadric of the net; therefore ψ_4 which passes through the intersection of ψ_3 and ψ is a quadric of the net. ψ_1, ψ_2, ψ_3, ψ_4 are therefore four quadrics of a net and the result follows.

We obtain also the following important theorem of plane geometry.

If there be four conics s_1, s_2, s_3, s_4 such that the four points of intersection of s_1, s_2 and the four points of intersection of s_3, s_4 lie on a conic s, the same is true of any other two pairs chosen out of the four conics.

Through s_1, s_2, s_3 describe any three quadrics ψ_1, ψ_2, ψ_3. These will define a net. A quadric ψ of the net can be drawn through one of the intersections of s_1, s_2 and one of the intersections of s_3, s_4. It will therefore contain, besides the eight points common to ψ_1, ψ_2, ψ_3, another point common to ψ_1, ψ_2 and thus the whole intersection of ψ_1, ψ_2. Hence ψ contains the four intersections of s_1, s_2 and one intersection of s_3, s_4; therefore it contains the conic s. Now through the intersection of ψ and ψ_3 draw a quadric ψ_4 to pass through any given point of s_4. This quadric cuts the plane in a conic having five points common with s_4 and therefore identical with s_4.

Four such conics s_1, s_2, s_3, s_4 are therefore the intersections of four quadrics of a net by a plane. The theorem is then obvious.

293. Conjugate points with regard to a net of quadrics. If two points P, P' are conjugate with regard to three quadrics ψ_1, ψ_2, ψ_3 of a net, not belonging to the same pencil, they are conjugate with regard to all quadrics of the net. For if ψ_4 be any other quadric of the net, we have seen by the above that a quadric ψ exists belonging to both pencils (ψ_1, ψ_2) and (ψ_3, ψ_4). If P, P'

are conjugate with regard to (ψ_1, ψ_2) they are also (by Art. 283) conjugate with regard to ψ. Also ψ, ψ_3, ψ_4 are quadrics of a pencil. Hence P, P' being conjugate with regard to ψ, ψ_3, they are also conjugate with regard to ψ_4.

Thus to every point P of space corresponds a point P' conjugate to P with regard to the net. P' is obtained as the intersection of the polar planes of P with regard to any three quadrics of the net, not belonging to the same pencil. Hence the polar planes of P with regard to the quadrics of the net pass through a fixed point P'.

294. Web of quadrics. A **web** of quadrics is the system of quadrics touching seven fixed planes. Reciprocating the properties of a net of quadrics we obtain the following:

The quadrics of a web touch an eighth fixed plane. If four quadrics ψ_1, ψ_2, ψ_3, ψ_4 belong to a web the common tangent planes to ψ_1 and ψ_2 and the common tangent planes to ψ_3 and ψ_4 all touch a quadric ψ.

To every plane of space there is one plane conjugate with regard to a web of quadrics, *i.e.* the poles of a fixed plane with regard to the quadrics of a web lie on another fixed plane.

In particular if the given plane be taken at infinity the locus of centres of quadrics of a web is a plane.

295. Any two quadrics may be transformed into one another by reciprocal polars. Two quadrics ψ_1, ψ_2 which have a proper common self-polar tetrahedron may be transformed (Art. 281) into coaxial quadrics ψ_1', ψ_2' by a homographic transformation. If a_1, b_1, c_1 and a_2, b_2, c_2, be corresponding semi-axes of ψ_1', ψ_2', construct a coaxial quadric ψ' with corresponding semi-axes a, b, c, where $a^2 = \pm a_1 a_2$, $b^2 = \pm b_1 b_2$, $c^2 = \pm c_1 c_2$. On reciprocation with respect to ψ', the vertices of ψ_1' transform into the tangent planes at the vertices of ψ_2' and the tangent planes at the vertices of ψ_1' into the points of contact of the corresponding tangent planes to ψ_2'. The reciprocal of ψ_1' is therefore a quadric ψ_1'' having the same vertices as ψ_2' and touching the same planes at those points; any principal plane thus intersects ψ_1'' and ψ_2' in conics which touch at four points, and therefore coincide. Since ψ_1'', ψ_2' intersect in three different conics they must coincide altogether; hence ψ_1', ψ_2' are polar reciprocals with respect to ψ'. On reversing the homographic transformation ψ' is transformed into a quadric ψ with respect to which ψ_1, ψ_2 are polar reciprocals. Owing to the alternatives of sign there are in all eight such quadrics ψ.

If we now consider the case where $a^2 = +a_1a_2$ and take A, A_1, A_2 the vertices of ψ', ψ_1', ψ_2' on the same side of the common centre O, then A_1 transforms by reciprocal polars with regard to ψ' into the tangent plane at A_2 and conversely. There are four quadrics ψ' for which this is the case. If O is now removed to a large distance, A, A_1, A_2 remaining accessible, A approaches the middle point of A_1A_2 and ψ', ψ_1', ψ_2' approach paraboloids having A_1A_2 as their common accessible axis and having common accessible principal planes. Thus two such paraboloids ψ_1', ψ_2' are reciprocal polars with respect to each of four coaxial paraboloids ψ'.

If we start with two quadrics ψ_1, ψ_2 touching at a single point B, these quadrics do not have a proper common self-polar tetrahedron, but two of the vertices of their common self-polar tetrahedron coincide at B, and the remaining vertices C and D lie in the common tangent plane at B, and are, in general, distinct from B. If we now apply a homographic transformation in which B, C, D are transformed into points at infinity in mutually perpendicular directions, ψ_1, ψ_2 are transformed into coaxial paraboloids ψ_1', ψ_2' and there are four paraboloids ψ' with respect to which ψ_1', ψ_2' are reciprocal polars. Transforming back there are four quadrics ψ with respect to which the given quadrics ψ_1, ψ_2 are reciprocal polars.

Similarly other cases where more than two of the vertices of the common self-polar tetrahedron coincide may be regarded as limits of a more general case; without going into details, we may expect that in such cases there will be at least one quadric with respect to which ψ_1, ψ_2 are polar reciprocals, this being the limit of one or more such quadrics in the more general case.

296. Quadrics outpolar and inpolar to a quadric. If there be one tetrahedron $ABCD$ inscribed in a quadric ψ_1 and self-polar for another quadric ψ_2, it will now be shown that there must exist any number of tetrahedra $A'B'C'D'$ inscribed in ψ_1 and self-polar for ψ_2; one vertex A' may be arbitrarily selected on ψ_1 and a second vertex B' may be any point of ψ_1 which is conjugate to A' for ψ_2.

Let α be the polar plane of A with respect to ψ_2, that is, the plane BCD; α meets ψ_1, ψ_2 in conics k_1, k_2 and BCD is clearly a triangle inscribed in k_1 and self-polar for k_2. Thus k_1 is outpolar to k_2.

If now A' is an arbitrary point of ψ_1, let L be a point in which the polar plane α' of A' with respect to ψ_2 meets k_1. Since L is a point of k_1, there exists a triangle LMN inscribed in k_1 and self-polar for k_2. The tetrahedron $ALMN$ is then inscribed in ψ_1

and self-polar for ψ_2. The plane AMN is the polar plane of L with respect to ψ_2 and it passes through A', since L, A' are conjugate for ψ_2. If this plane meets ψ_1, ψ_2 in conics s_1, s_2, then, as before, s_1 is outpolar to s_2, and there exists a triangle $A'M'N'$ inscribed in s_1 and self-polar for s_2, so that the tetrahedron $A'LM'N'$ is inscribed in ψ_1 and self-polar for ψ_2.

The plane $LM'N'$ is then identical with α'. If it meet ψ_1, ψ_2 in conics k_1', k_2' then, again, k_1' is outpolar to k_2'; any point B' of ψ_1 conjugate to A' for ψ_2 is a point of k_1' and is one vertex of a triangle $B'C'D'$ inscribed in k_1' and self-polar for k_2'. The tetrahedron $A'B'C'D'$ is then inscribed in ψ_1 and self-polar for ψ_2.

A quadric ψ_1 which is such that there are tetrahedra inscribed in ψ_1 and self-polar for another quadric ψ_2 is said to be **outpolar** to ψ_2.

In a similar manner we can show that, if one tetrahedron is circumscribed to ψ_1 and self-polar for ψ_2, an infinity of such tetrahedra can be constructed, one face α of which can be taken as any tangent plane to ψ_1 and a second face β is any tangent plane to ψ_1 conjugate to α for ψ_2.

The quadric ψ_1 is then said to be **inpolar** to ψ_2.

If there be a tetrahedron $ABCD$ inscribed in a quadric ψ_1 and self-polar for a quadric ψ_2, so that ψ_1 is outpolar to ψ_2, let ψ be a quadric with respect to which ψ_1, ψ_2 are polar reciprocals (Art. 295). Reciprocating with respect to ψ, $ABCD$ is transformed into a tetrahedron $\alpha\beta\gamma\delta$ circumscribed to ψ_2 and self-polar for ψ_1. Thus, if ψ_1 is outpolar to ψ_2, then ψ_2 is inpolar to ψ_1.

297. Outpolar and inpolar envelopes and loci of two quadrics. In general, a plane does not cut two quadrics ψ_1, ψ_2 in conics k_1, k_2 such that k_1 is outpolar to k_2. This may, however, happen in certain cases.

Let l be any line, P a point where it meets ψ_1, π the polar plane of P with regard to ψ_2, meeting l at R. Then π meets ψ_1, ψ_2 in conics s_1, s_2.

Let s be the harmonic envelope (Art. 201) of s_1, s_2. From R draw two tangents r, r' to s, determining planes $\rho = Pr$, $\rho' = Pr'$, which pass through l.

If r meet s_1 at U, V then, by the property of the harmonic envelope U, V are conjugate for s_2 and therefore for ψ_2. Since U, V lie in π, they are conjugate to P for ψ_2, so that PUV is self-conjugate for ψ_2 and is inscribed in the conic k_1 in which ρ meets ψ_1. If ρ meets ψ_2 in the conic k_2, then k_1 is outpolar to k_2. Similarly the conic k_1' in which ρ' meets ψ_1 is outpolar to the conic k_2' in which ρ' meets ψ_2.

Thus through any line l two planes can be drawn, meeting the quadrics in conics satisfying the given conditions. The planes for which this holds therefore envelop a quadric ψ which may be termed the **outpolar envelope** of ψ_1 with respect to ψ_2. Clearly there will be a second envelope of planes meeting the quadrics in conics k_1, k_2 such that k_2 is outpolar to k_1. This may be called the **inpolar envelope** of ψ_1 with respect to ψ_2.

If, in the above, ψ_2 is a cone, and l passes through the vertex of the cone, π meets ψ_2 in a line-pair s_2. In this case the harmonic envelope s is the product of ranges on the lines of s_2 conjugate for s_1 and the tangents r, r' coincide with the line-pair s_2. In this case the tangent planes through l to ψ coincide with the tangent planes through l to ψ_2, so that ψ_2 is a tangent cone to ψ.

Reciprocating the above theorems, we have :

The locus of the vertex of a three-edge whose faces touch ψ_1, and which is self-conjugate for ψ_2, is a quadric ψ, the **inpolar locus** of ψ_1 with respect to ψ_2, or the **outpolar locus** of ψ_2 with respect to ψ_1.

If ψ_2 degenerates into a conic, this conic lies on ψ.

A very important case of the latter is when ψ_2 is the circle at infinity. The inpolar locus of ψ_1 with respect to $\odot$ is then a quadric containing $\odot$, that is, a sphere. This sphere is the locus of the intersection of three tangent planes to ψ_1 which are mutually conjugate for $\odot$, that is, mutually perpendicular. This is known as the **orthoptic sphere** of ψ_1.

Since tangent planes at the extremities of a diameter are parallel, to every point of the orthoptic sphere corresponds a second point, symmetrically situated with respect to the centre of the quadric. The latter is therefore also the centre of the orthoptic sphere.

Examples

1. Prove that if V is a point on the outpolar locus of ψ_1 with respect to ψ_2, any number of three-edges with V for vertex exist, whose edges touch ψ_1 and which are self-conjugate for ψ_2.

2. If ψ_1, ψ_2 are polar reciprocals with respect to a quadric ψ, the inpolar locus of ψ_2 with respect to ψ contains the intersection of ψ and ψ_1.

3. Prove that the outpolar envelope of a quadric ψ_1 with respect to another quadric ψ_2 touches the tangent plane to ψ_2 at any point P of the intersection of ψ_1, ψ_2.

State the corresponding property for the outpolar locus of ψ_1 with respect to ψ_2.

4. In any transformation by reciprocal polars which transforms ψ_1 into ψ_2, the inpolar locus of ψ_2 with respect to ψ_1 reciprocates into the outpolar envelope of ψ_1 with respect to ψ_2.

298. Pencil of quadrics outpolar to a quadric. If two quadrics ψ_1, ψ_2 of a pencil are outpolar to the same quadric ψ, every quadric of the pencil is outpolar to ψ.

For let ψ_3 be any other quadric of the pencil, A any point of the twisted quartic q which defines the pencil. The polar plane α of A with regard to ψ will meet ψ_1, ψ_2, ψ_3, ψ in conics k_1, k_2, k_3, k. The conics k_1, k_2, k_3 belong to the same pencil in α. But k_1, k_2 are both outpolar to k. Hence, by Art. 232, every conic of the pencil is outpolar to k, and therefore k_3 is outpolar to k. We can therefore find a triangle BCD inscribed in k_3, which is self-polar for k, and therefore self-conjugate for ψ. Hence the tetrahedron $ABCD$, which is clearly inscribed in ψ_3, is self-polar for ψ, which proves what is required.

As a particular case, if the twisted quartic breaks up into two conics, the plane pair through these conics is conjugate for ψ.

If the quadrics ψ_1, ψ_2 are spheres outpolar to the same quadric ψ, the last-mentioned case arises, the plane-pair being the common radical plane and the plane at infinity. The common radical plane then passes through the centre C of ψ. Thus the sphere σ of centre C, whose radius is the distance from C to the limiting points of the pencil (ψ_1, ψ_2) is orthogonal to all the spheres of this pencil which are necessarily spheres outpolar for ψ.

In particular the point-spheres of the pencil are the vertices of spherical cones outpolar for ψ, that is, such that ψ is inscribed in three-edges self-polar for these spherical cones; these are tri-rectangular three-edges, so that the point-spheres in question lie on the orthoptic sphere of ψ. This orthoptic sphere is therefore orthogonal to every sphere of the pencil; and since ψ_1, ψ_2 were arbitrarily selected in the first instance, the orthoptic sphere of ψ is orthogonal to every sphere outpolar to ψ.

EXAMPLES XVI

1. Prove that the tangent planes at the vertices of a twelve-face eight-point inscribed in a quadric form a twelve-point eight-face having the same diagonal tetrahedron.

2. If two non-parallel circular sections of a quadric are given, show how to find the directions of its axes.

The circles circumscribed about two triangles ABC, DEF, in different planes, have a common diameter with the same extremities M, N; and O

MISCELLANEOUS EXAMPLES

1. Prove that if, in two collinear projective ranges, the self-corresponding points coincide at O, and if P, P' be any other pair of corresponding points, then

$$\frac{1}{OP'} - \frac{1}{OP} = \text{const.}$$

OA, OB, OC are three rays through O, and the correspondence between rays OP, OP' is defined as follows: if OQ is taken harmonically conjugate to OP with respect to OA, OB, then OP' is harmonically conjugate to OQ with respect to OA, OC. Show that the pencils $[OP]$, $[OP']$ are projective and that they have OA for a double self-corresponding ray.

2. Two conics k, k' have three-point contact at O, and through O a ray is drawn meeting the conics again at T, T'. The tangents at T, T' meet the common tangent at O in P and P'. Prove that $[P]\barwedge[P']$ and that the self-corresponding points coincide at O.

What happens if the conics have four-point contact?

3. Give a method of constructing a square whose sides taken in order shall pass through four given points A, B, C, D. Perform the construction, having given that $AB=2$ inches, $BC=CD=2{\cdot}5$ inches, $DA=1{\cdot}5$ inches, and the angle $BAD=120°$.

4. Show how to cut the hyperbola whose equation in Cartesian co-ordinates is $\frac{x^2}{4} - \frac{y^2}{9} = 1$ from a right circular cone.

5. ABC is an equilateral triangle of side 2 inches, lying in a horizontal plane: the inscribed circle touches the sides AB, AC at D, E respectively. From the point V, one inch vertically above A, the figure is projected upon the plane through A parallel to the plane VBC. Draw the projection of the triangle ADE, and find the vertex and focus of the projection of the circle.

[For clearness it is advisable to rabat through the obtuse angle.]

6. Draw a triangle ABC with sides $BC=3$ inches, $CA=AB=4$ inches, and mark the middle point D of AB. A parabola touches the three sides of the triangle ABC and has its axis parallel to CD; construct (i) the tangent at the vertex, (ii) the axis, (iii) the directrix.

7. Given a conic k and a line p', not tangent to k, show how to find, by a geometrical construction, (i) the point of contact of p' with the conic k' which has four-point contact with k at a given point O and touches p', (ii) the other extremity of the diameter of k' through O, (iii) the orthoptic circle of k'.

8. If $ABCDEF$ ... be a closed polygon of any number of sides; P, Q, R, S,

T, ... points of section arbitrarily taken on AB, BC, CD, DE, EF, ... respectively; prove that the value of the continued product of ratios

$$\frac{AP}{PB}\cdot\frac{BQ}{QC}\cdot\frac{CR}{RD}\cdot\frac{DS}{SE}\cdot\frac{ET}{TF}\cdots$$

is unaltered by projection.

Hence show that, if P, Q, R, S, T, ... are collinear, the value of the above product is $+1$ or -1 according as the number of sides of the polygon is even or odd.

Is the converse of this theorem generally true?

9. $ABCD$ is a skew quadrilateral whose sides AB, BC, CD, DA are cut by a plane at P, Q, R, S respectively. Prove that

$$AP.BQ.CR.DS = PB.QC.RD.SA;$$

and conversely if P, Q, R, S be any four points on these sides, so chosen that the above relation holds, then P, Q, R, S, are coplanar.

Generalise the first part of this theorem for a skew polygon of more than four sides.

Prove that if a sphere touch each of the sides of a skew quadrilateral internally, the points of contact are coplanar.

10. Any two tetrahedra $ABCD$, $A'B'C'D'$ for which each pair of corresponding edges AB, $A'B'$ meet have the joins of corresponding vertices concurrent.

11. Show that the cross-axis of two coplanar projective ranges is parallel to the line joining their vanishing points.

12. Given a conic k and a line x, show how to construct the centre of the conic k' which touches k at a given point O, passes through a second given point A' not on k and meets x in the same two points (real or imaginary) as k.

13. The lines joining a point O to the vertices of a triangle ABC meet the opposite sides at D, E, F and the sides of DEF at P, Q, R. Prove that BC, EF, QR are concurrent at U, with a similar meaning for V and W; that U, V, W are collinear and that P, D are harmonically conjugate with respect to O, A.

Show also that, when O describes a fixed line l meeting BC at X, P describes a fixed line through X and UVW touches a fixed conic, which touches BC at the harmonic conjugate of X with respect to B, C.

14. Two tangents a, b to an ellipse are parallel to the axes and a touches the ellipse at A. P_1P_2, P_2P_3, ... are a number of equal segments on a. From P_1, P_2, etc. tangents are drawn touching the ellipse at T_1, T_2, etc., respectively, and AT_1, AT_2, ... meet b at Q_1, Q_2, ... respectively.

Show that the segments Q_1Q_2, Q_2Q_3, ... are all equal.

15. Prove that if a variable tangent to a parabola meets two fixed tangents at P and Q, the locus of the middle point R of PQ is a straight line.

16. A line AP drawn from the vertex A of a triangle to a point P in the opposite side BC is divided at Q so that $PQ : QA = BP : PC$. Prove that the locus of Q is a parabola which passes through B, touches AC at A and has its axis parallel to BC.

17. Prove that, if P be a variable point on the fixed line l, C and E any fixed points, b and d any fixed lines, then the conic passing through the following five points: (1) P; (2) bd; (3) (CE, b); (4) E; (5) (CP, d), passes through a fixed point on l.

18. Prove that if a conic touch the sides BC, CA, AB of a triangle ABC at D, E, F respectively, and if AD, BE, CF meet the conic again at P, Q, R respectively, the tangents at P, Q, R meet BC, CA, AB respectively at three collinear points.

19. Show that it is possible to construct a conic circumscribing a triangle ABC and touching the parallels through A, B, C to the opposite sides.

20. Obtain a straight line construction for finding the fourth point of intersection of two conics which pass through three points A, B, C, without drawing the conics, two other points on each conic being given.

21. From a given point O on a conic k, any two chords OP, OQ are drawn, and through their other extremities P, Q two chords PR, QS are drawn parallel to OQ, OP respectively. Prove that RS is parallel to the tangent at O.

22. Show that any two points A, B of the plane and the four points of contact of tangents from A, B to any conic k in the plane are six points of a conic.

State and prove the reciprocal result.

23. Prove that conjugate ranges with respect to a conic s, whose bases a, b intersect on s, are perspective from the pole of the chord joining the other intersections of a, b with s.

Prove that conjugate pencils with respect to a conic s, whose vertices A, B lie on a tangent to s, are perspective, the axis of perspective being the polar of the meet of the other tangents from A, B to s.

If a point A of a line x is conjugate to more than one other point of that line, x must be a tangent to the conic and A its point of contact.

24. A conic touches the sides BC, CA of a triangle ABC at points D, E respectively, and meets AB at two points X, Y. The tangents at X, Y meet at T, and $U=(XD, YE)$, $V=(XE, YD)$. Prove that each of the triangles DEU, DEV is in perspective with the triangle ABC; and that T, U, V lie on the same line through C.

Show further that, if M, N be the meets of BC, CA with the tangents at X, Y respectively, the lines AB, DE, MN are concurrent.

25. Prove that, if two conics k_1, k_2 touch at O and meet at two other distinct points, and if through O any line be drawn to meet k_1, k_2 again at P, Q respectively and R is harmonically conjugate to O with respect to P, Q, then the locus of R is a conic touching k_1, k_2 at O.

26. If l, m be a variable pair of perpendicular lines conjugate with respect to a conic k whose centre is O, and l pass always through a fixed point P, prove that in general m always touches a fixed parabola, of which the axes of k and the polar of P for k are tangents.

Show further that the tangents from P to this parabola bisect the angles between the tangents from P to k, and that OP is the directrix of the parabola.

27. If in Fig. 26 AA', BB, CC' meet at a point, show that the Pascal line is the polar of this point.

28. Two triangles are inscribed in a conic. The sides of the one meet the sides of the other in nine points. Show that the join of any two of these nine points is a Pascal line of the six vertices of the triangles, unless it is one of the sides of the triangles.

29. A variable tangent to a conic k meets two fixed perpendicular tangents a, b at P, Q respectively; and the perpendiculars to a, b at P, Q meet at R. Prove that, if k is a central conic, the locus of R is a rectangular hyperbola

whose asymptotes are the tangents of k parallel to a, b; and that this hyperbola passes through the points of contact of a and b with k.

Investigate also the locus of R when k is a parabola.

30. Given a focus of an ellipse, one point on the curve and one tangent (not at the given point show that the locus of the second focus is one branch of a hyperbola.

Construct the asymptotes of this hyperbola and find the least possible length for the major axis of the ellipse.

31. The tangents to an ellipse at points P, Q meet at T, and S is a focus of the ellipse; the circle on ST as diameter meets TP, TQ again at Y, Z respectively. Prove that ST is bisected by the diameter of the ellipse perpendicular to YZ.

Show that, if T describes the orthoptic circle of the ellipse, the middle point of YZ describes a third circle, whose radius is half that of the orthoptic circle; and that the envelope of YZ is the conic which has one focus at the centre of the ellipse, and the third circle for auxiliary circle.

32. Show that the common self-polar triangle of two circles which do not intersect in real points is formed by the limiting points of the two circles and the point at infinity on their radical axis.

33. Prove that if in two coplanar projective figures two pairs of corresponding pencils are perspective, the same line being the axis of perspective in both cases, then the two figures are altogether in plane perspective.

34. Show that the transformation by reciprocal polars is the only reciprocal transformation in the plane for which every point P of the plane has the same line for its reciprocal in the two figures.

35. If through the in-centre K of a triangle ABC, lines KL, KM, KN be drawn, perpendicular to KA, KB, KC respectively, and meeting the opposite sides BC, CA, AB at L, M, N respectively, prove that L, M, N lie on a straight line which is the radical axis of K and the circle through A, B, C.

36. A central conic s has foci F, F' and corresponding directrices, f, f'. A variable line l through F meets f' at Q and s at R, R'; and P is harmonically conjugate to Q with regard to R, R'. Prove that the locus of P is a conic k, which passes through F, F' and belongs to the pencil determined by s and its orthoptic circle, and that the same conic k arises in this manner from lines through F'.

If $F'P$ meet f at Q', prove that the envelope of QQ' is a conic c confocal with s and touching f, f'; and that k, c are reciprocal with respect to s.

37. A conic inscribed in a triangle has one focus at the circumcentre. Show that the other focus is at the orthocentre and that the length of the major axis is equal to the radius of the circumcircle.

[Use Ex. 35, reciprocating with respect to a circle centre K.]

38. Four conics pass through three given non-collinear points A, B, C and have a fourth given point S for focus. Prove that the directrices which correspond to S meet in pairs on the sides of the triangle ABC and form a quadrilateral of which ABC is the diagonal triangle.

[Reciprocate with regard to S: the corresponding property of the inscribed and escribed circles of a triangle is obvious.]

If one of the four conics is a circle show that the directrices of the other three, which correspond to S, are the joins of the mid-points of the sides of the triangle ABC.

39. Show that the inverse of the cubic curve

$$x(x^2+y^2)=cy^2$$

with respect to a circle whose centre is the origin O is a parabola with its vertex at O.

Derive the following properties of this cubic curve by inversion from the corresponding properties of the parabola :

(i) If the circle which touches the cubic curve k at any point P and passes through O meets the axis of y again at Q, the circle through O and Q orthogonal to this circle meets the axis of x again at a fixed point R.

(ii) If the circles through O which touch k at points P_1, P_2 are orthogonal, their second intersection lies on a fixed circle through O ; and O, P_1, P_2, R are concyclic.

40. If two tangents to a parabola make equal angles with a fixed straight line, show that the chord of contact must pass through a fixed point.

41. The three diagonals of a complete quadrilateral are divided harmonically at P, P' ; Q, Q' ; R, R' in any manner. Show that these six points lie on a conic.

42. A fixed line l lies in the plane of a conic k ; and d is the diameter of k conjugate to l. Through each point P of l a line p is drawn perpendicular to the polar of P with respect to k. Prove that, when l is not a diameter of k, these perpendiculars p all touch a certain fixed parabola, which touches l and has its axis perpendicular to d. Show also that d is the directrix of this parabola.

43. Prove that the in- and ex-centres of a triangle self-polar for a rectangular hyperbola lie on the curve.

44. Prove that the tangent at a point P of a parabola and the common chord of the parabola and its circle of curvature at P are equally inclined to the axis of the parabola.

The normals to a parabola at P, Q, R are concurrent, and the poles, with respect to the parabola, of the common chords of curvature at P, Q, R are X, Y, Z respectively. Prove that the perpendiculars from X, Y, Z to their respective polars are concurrent.

45. A family of conics have double contact with a circle of centre C at two fixed points A, B. Prove that the foci of such conics on their axes parallel to AB and the points of contact of tangents from C to the conics all lie on the circle circumscribing ABC.

[Project the quadrangle $AB\Omega\Omega'$ into $\Omega\Omega' AB$; Ω, Ω' being the circular points at infinity in the plane.]

46. A triangle ABC is circumscribed to a conic k, the base AB being of given length and lying in a fixed tangent t to k. Show that the locus of the vertex C is a conic having four-point contact with k at the point of contact E of the tangent to k parallel to t.

47. If two tangents to a parabola make a constant angle (other than a right angle) with each other, show that the locus of their intersection is a hyperbola and that their chord of contact envelops a conic.

48. A variable conic k touches the sides of a triangle ABC and a fourth fixed straight line l. Prove that the locus of P, the point of concurrence of the lines joining the vertices of ABC to the points of contact of k with the opposite sides, is a conic s through A, B, C and that the range $(ABCP)$ on s is homographic with the range determined on any tangent to k by BC, CA, AB and l.

49. If an involution pencil of the first order is homographic with a simple pencil of the second order, show that their product is in general a curve of the fifth degree, having the vertex of the involution pencil for a multiple point of the fourth order.

Prove also that every quintic with a quadruple point can be described in this manner.

50. Tangents are drawn from a fixed point to each member of a range of conics. Prove that the locus of the points of contact is a quintic curve.

51. Prove that if an involution of points on a conic is homographic with an involution pencil whose vertex is not on the conic, there are in general six points of the conic which lie in one of their corresponding lines.

Hence show that, in general, the product of two homographic involution pencils of the first and second orders respectively is a sextic curve, the vertex of the pencil of first order being a quadruple point.

52. Show that the product of an involution pencil of the second order and a simple pencil of the first order, homographic with the first-named pencil, is a quartic curve, of which the vertex of the second-named pencil is a double point.

53. If a conic s_1 be triangularly inscribed in another conic s_2, prove that the tangent to s_1 at a common point of s_1, s_2 passes through the point of contact with s_2 of a common tangent of s_1, s_2.

54. The centre of a circle lies on a rectangular hyperbola and the centre of the hyperbola lies on the circle. Show that the polar reciprocal of the hyperbola with respect to the circle is a parabola, of which the centre of the hyperbola is the focus, and prove that any two of these three curves are polar reciprocals with regard to the third one.

55. Prove that the circle, hyperbola and parabola of the last example are such that any one of the three is both outpolar to, and triangularly circumscribed about, any other.

56. If A, B be two fixed points on a sphero-conic, P a variable point on the curve, and if the arcs PA, PB meet a cyclic line at X, Y respectively, show that the arc XY is of constant length.

57. If through a point O on a sphero-conic two perpendicular great circle arcs be drawn, meeting the sphero-conic again at two points (not antipodal to each other or to O), the great circle arc through these two points passes through a fixed dyad on the sphere.

58. Show that the envelope of joins of homographic dyads on a sphero-conic is another sphero-conic, having double contact with the given sphero-conic, antipodal contacts being reckoned as one.

59. Prove that the equation of the circle at infinity in plane co-ordinates is

$$l^2+m^2+n^2=0.$$

60. In the case of a quadric of revolution show that the focal conics reduce to (1) a circle in the diametral plane perpendicular to the axis of revolution ; (2) two points F, F' called principal foci, symmetrically situated with regard to the centre of the quadric on the axis of revolution.

61. If a quadric touch the six edges of a tetrahedron, the lines joining each vertex of the tetrahedron to the pole of the opposite face are concurrent.

62. Show that the locus of the polar of a given line with respect to a system of confocal quadrics is a hyperbolic paraboloid, one of whose generators is the line at infinity in the plane perpendicular to the given line.

63. A wrap of planes $[\pi]^2$ of the second order in a star of vertex O is homographic with an axial pencil $[\pi^1]$, whose axis does not pass through O. If $[\pi]^2$, $[\pi^1]$ have a self-corresponding plane, show that their product is a regulus.

64. Prove that the product of a regulus and a homographic axial pencil is in general a twisted cubic.

65. State, and prove independently, the theorem reciprocal to that of Ex. 64.

66. Prove that if through a fixed point O lines be drawn each of which is perpendicular to its polar line for a given quadric, these lines generate a cone of the second order.

Show that lines through a fixed point which are normal to quadrics of a confocal system are generators of a cone of the second order.

67. P is a variable point on a fixed diameter of a quadric ψ, and R is the foot of the perpendicular from P upon its polar plane with respect to ψ. Show that the locus of R is a rectangular hyperbola.

68. Prove that the product of two homographic pencils of quadrics is, in general, a surface of the fourth order; and that the product of a pencil of quadrics with a homographic axial pencil is, in general, a cubic surface.

69. Prove that, if a regulus or a conical pencil of the second order is homographic with another regulus or conical pencil of the second order (not having the same vertex as the first conical pencil), then, in general, there are four rays, and four rays only, which intersect their corresponding rays.

70. Two stars with different vertices O_1, O_2 are reciprocally related. Show that the locus of the meet of a line of either star with its corresponding plane of the other is a quadric ψ passing through O_1, O_2, which meets any plane α in the locus of incident points of the reciprocal fields determined in α by the two stars.

71. If $[P]$, $[Q]$, $[R]$, $[S]$ be four projective ranges on four arbitrarily chosen straight lines, prove that there are in general four, and only four, planes, which contain four corresponding points P, Q, R, S.

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